Collaboration Between and Within Groups*

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Abstract

We study the ability of multi-group teams to undertake binary projects in a decentralized environment. The equilibrium outcomes of our model display familiar features in collaborative settings, including inefficient gradualism, inaction, and contribution cycles, wherein groups alternate taking responsibility for moving the project forward. Expected delay grows more than proportionally with project size, and some welfare-enhancing projects are not completed, even as agents become arbitrarily patient. A team composed of two equally large groups can complete larger projects than a fully homogenous team, even as the difference in preferences for completion among the two groups is arbitrarily small. Moreover, if the project is sufficiently large, the two-group team always completes the project strictly faster.

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1 Introduction

In the textbook notion of a team, agents work together to attain a common goal. In many real life scenarios, however, teams are composed of distinct groups. In these contexts, agents have to collaborate with both peers and "outsiders," who can have a different valuation for completion of the project, or a different effectiveness in moving the project forward. In a research lab, for instance, senior scholars collaborate with junior researchers, for whom a publication can have a larger career impact. In corporations, new product launches usually involve both Product and Operations personnel, who jointly develop new infrastructure to make the product available to clients. In open source software development, both amateur and professional developers contribute towards writing the code that drives projects like Linux and Mozilla Firefox. Collaboration between groups is also paramount in applications within political economy. In lobbying, beneficiaries of a piece of legislation who seek to influence legislators typically belong to different industries, each of which has a different stake in the bill's success.

In this paper, we study the ability of such multi-group teams to "get things done." In particular, we are interested in understanding how heterogeneity in costs or preferences across groups shapes the expected time to complete projects, as well as the size of projects that the team can complete. On the behavioral (positive) side, we analyze how heterogeneity across groups affects the pattern of contributions by members of the two groups. Does the group with a higher valuation for completion of the project contribute more? Do they contribute in earlier or later phases of the project? We study these questions in a decentralized environment, without commitment. By this we mean that the identity of the contributor and the amount of the contribution at each point in time are not determined by a central authority or contract, but instead come from random opportunities to add value (e.g. ideas), and individual incentives to cooperate. This is particularly relevant in environments in which effort is elicited through a horizontal structure, as in open source contributions or cross-functional teams in firms. In line with our examples, we treat projects as binary; that is, as *threshold* public goods (i.e., payoffs accrue only when the paper is completed, when the product is launched, when the bill passes, etc).

In the model, two groups of internally identical agents – say junior and senior researchers– collaborate on a binary project. To capture the decentralized nature of contributions, we assume that in each period an agent is selected at random to make a contribution to the project. We allow juniors and seniors to have a different probability of having an opportunity to contribute, but assume that this is equal among agents of the same group. In line with the binary project assumption, we assume that agents of group m get a payoff $\theta^m > 0$ when the project is completed, and zero otherwise. To distinguish strategic considerations from technological incentives for gradualism, we assume that the cost of contributing is linear, and assume there are no deadlines, or asymmetric information among agents. To maintain within-group homogeneity, we focus on group-symmetric subgame perfect equilibria (SPE).

We show that the game has a generically unique equilibrium in pure strategies. The equilibrium has the feature that agents' strategies are history-independent (i.e., markovian), conditioning only on the remaining amount of effort s required to complete the project. Exploiting the markovian structure of equilibrium, we show that the game can be solved recursively from a subgame starting in a state $s = \phi_1 > 0$, the largest state at which members of one group, say juniors, finish the project outright. We call this subgame a *small project*.

Our first main result comes from the characterization of the final small project, which juniors complete without delay. We show that in equilibrium, seniors are only willing to only make partial contributions towards completion, moving the project forward in a series of steps, which decrease in size as the project gets farther from completion. Remarkably, the size of these contributions converges to zero at some point $\overline{s}_1 < \phi_1$. This means that there is a set of states $(\overline{s}_1, \phi_1]$ in which seniors do nothing, fully free-riding on juniors for the completion of the project. This *inaction* by seniors implies that at ϕ_1, the project is stalled with positive probability, and only moves forward (and in fact is completed outright) when a junior member has an opportunity to play.

Seniors' slugishness to move the project forward from states $s < \overline{s}_1$ is due to in-group free riding. Their inaction at ϕ_1, on the other hand, is due to out-group incentives. As the number of steps and the associated delay introduced by senior agents increases, juniors' willingness to contribute *increases* as well. For them, the more sluggish seniors become, the more attractive completion becomes in order to avoid costly delay. Formally, we show that for any additional step in senior members' strategy, the maximal state ϕ_1 at which juniors are willing to complete the project outright increases by a larger amount than the increase in the seniors' cutpoint. This assures the existence of a set of states in which only juniors contribute to advance the project.

Our second main result builds on the fact that the equilibrium of the game has a recursive structure from ϕ_1 on. We show that the overall game is strategically equivalent to a sequence of small projects $\{\phi_\tau\}$. As in the case of the final small project, each small project is completed exclusively and at once by members of one active group, while members of the other group stay put. As a result, the equilibrium has two different sources of delay (on path):

inaction and *gradualism*. The project is partitioned into multiple steps (small projects) precisely due to in-group free-riding (i.e., because each group is composed of multiple members). In fact, if "groups" are composed of single agents, the project is completed in two large steps, as in Compte and Jehiel (2003).[1] With multi-member groups, instead, incentives to contribute are never extinguished in one-shot, but rather conserved by the existence of a dependable peer on whom an agent of the active group can free ride. The size of each small project is thus reduced by in-group free riding as the project is farther from completion (in the initial stages of the project).

In our third main result for the characterization of equilibrium, we turn to how effort is divided among groups throughout the project. We show that in sufficiently large projects, the combination of inaction and gradualism leads to endogenous *contribution cycles*, wherein juniors and seniors alternate taking responsibility for moving the project forward, while members of the other group do nothing (Figure 1). This alternation of effort on the path of play emerges since "owning" a small project is costly. Thus, the continuation value of the group that anticipates being active contributors in the future decreases faster, as the project moves farther away from completion, than that of the passive group.

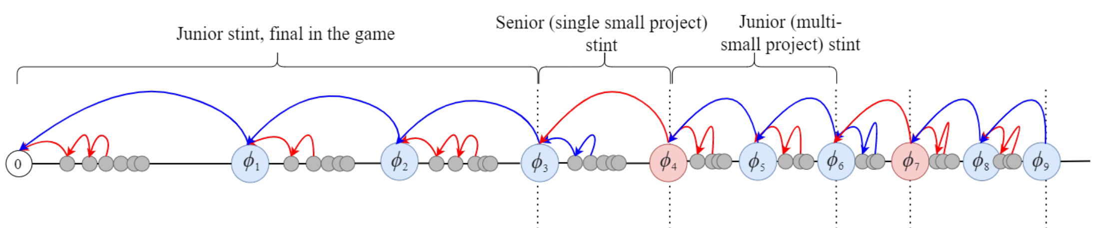


Figure 1: Junior (blue) and senior (red) agents alternate contributing in stints of small projects towards completion of the project.

Different from the one-to-one alternation typically assumed in the literature, these contribution cycles are generally asymmetric, with one group taking on stints of multiple consecutive projects at a time, while members of the other group only undertake a single small project at a time. Crucially, relative preference intensity or cost differentials among groups only affect the identity of the group which finishes the project, but do not affect contribution cycles in earlier phases of the game. This is because in equilibrium, the last contribution stint

[1] With only two identical members (two single-member "groups" without heterogeneity), and linear costs, Admati and Perry (1991) show that there exist two equilibria, with and without gradualism. Compte and Jehiel (2003) show that if the two individuals have different valuation for completion of the project, only the non-gradual equilibrium survives. Thus, in the model of Admati and Perry, gradualism requires full homogeneity or strictly convex costs (see also Kessing (2007)). Other explanations for gradualism include history-dependent outside options (Compte and Jehiel (2004)), history-dependent strategies (Marx and Matthews (2000)) or moral hazard (Bonatti and Hörner (2011), Giorgiadis (2014)).

by juniors is precisely long enough to eliminate all differences in willingness to pay across groups due to preference intensity. As a result, contribution cycles are determined solely by free-riding incentives. In particular, we show that the group of agents who are active contributors in stints of multiple consecutive small projects is the group for which in-group free-riding incentives are *larger* (i.e., the largest group), and out-group free-riding incentives are *smaller* (i.e., the group which is more likely to have opportunities to contribute).[2]

As it is clear by now, equilibrium outcomes are inefficient due to the delay caused by gradualism *and inaction* along the path of play. In the paper, we show that expected delay grows more than proportionally with project size, and goes to infinity as the project size approaches the maximum level that is completed in equilibrium. But delay is not the only manifestation of inefficiency. In fact, we show that some welfare-enhancing projects are not completed. What's more, due to in-group free riding, this is true even as agents become arbitrarily patient.

A natural question is how this inefficiency compares with the case of no heterogeneity across groups. We show that across-team heterogeneity increases the size of the largest implementable project by an amount bounded away from zero, for any fixed discount factor δ and group size n. As the size of the group grows large, this benefit decreases and becomes zero in the limit. In terms of delay, we show that for large n and δ, projects of relatively small size are completed faster by homogenous teams. As the total project size increases, however, eventually the two-group team always dominates in performance.

Our results speak to an influential observation by Mancur Olson in his landmark work *The Logic of Collective Action* (1965, p. 35):

> This suboptimality or inefficiency [in provision of a collective good] will be somewhat less serious in groups composed of members of greatly different size or interest in the public good. In such unequal groups, on the other hand, there is a tendency toward an arbitrary sharing of the burden of providing the collective good. The largest member, ... who would on his own provide the largest amount of the good, bears a disproportionate share of the burden of providing the collective good.

Olson's suggestion that inefficiency in public good provision would attenuate with greater within-team heterogeneity, is borne out by our analysis. We show that a team composed of two equally large groups can complete larger projects than a fully homogenous team, even as the difference in preferences for completion among the two groups is arbitrarily small. In

[2] In the special case in which both groups have equal in-group and out-group free-riding incentives, both groups undertake a single small project at a time.

addition, if the project is sufficiently large, the two-group team always completes the project strictly faster in expectation. Olson's intuition that *asymmetries* across the agents will lead to work being shared unequally is also reflected in our model locally, since only a subset of the agents bear the burden of each small project. However, we show that in the context of multi-group teams, the overall share of work within the project borne by a given group m is not determined only (or primarily) by preference intensity, as Olson suggests, but by in-group and out-group free-riding incentives.

The remainder of the paper is structured as follows. Section 2 provides a review of the related theoretical and experimental literature on inter-group collaboration within teams, particularly in public goods provision. In Section 3, we outline the model in detail. Section 4 is the heart of the paper's analysis and contains all main results. Section 4.1 derives equilibrium behavior within each "small project," while Section 4.2 builds on this work to characterize the equilibrium structure in large projects. Section 5 concludes. All proofs are in the Appendix (proofs of Lemma 4.1 and Theorem 4.9 are in the Online Appendix).

2 Related Literature

Our paper is related to the literature on dynamic contributions to public goods. This literature studies, in particular, how free-riding incentives are changed by dynamic considerations.

Two early papers are Fershtman and Nitzan (1991) and Admati and Perry (1991). Fershtman and Nitzan (1991) consider a problem in which the public good delivers continuous benefits, and study the efficiency of public good provision. Admati and Perry (1991), instead, consider a binary project, in which benefits are attained only when the project is completed, as in our paper. The distinction between continuous and binary public goods is fundamental for dynamic free-riding incentives. As pointed out by Kessing (2007), in continuous public goods, efforts are *strategic substitutes* across time. In this case, contributions lead to a reduction of effort in the future. In binary public goods, instead, efforts are *strategic complements* across time, in that a higher contribution by one agent induces larger contributions in the future.[3] The latter is a basic feature of our model, as well as of Admati and Perry (1991), Compte and Jehiel (2003, 2004)), Kessing (2007), Giorgiadis (2014), and Bowen, Georgiadis, and Lambert (2016). On the other hand, Battaglini, Nunnari, and Palfrey (2014) and Harstad (2016), consider models in which the public good gives continuous benefits, as in Fershtman

[3]This induces inefficient delay due to a time-consistency problem, because contributing more heavily early on and then reducing effort is not credible. In order to prevent others from free riding, each player delays her contributions inefficiently.

and Nitzan (1991), while Marx and Matthews (2000) consider both cases.

Our paper is most closely related to Admati and Perry (1991) and Compte and Jehiel (2003). Admati and Perry (1991) consider a model in which two identical agents take turns contributing to the public good. When the cost of contributing is strictly convex, the game has an essentially unique SPE path, in which agents make gradual contributions to the project.[4] Compte and Jehiel (2003) consider a variant of Admati and Perry's model in which agents differ in their valuation of the good and focus on the case of linear costs. In contrast to Admati and Perry's model, in equilibrium, each player makes only one large contribution. Moreover, whenever the project is socially desirable, it is completed as players grow arbitrarily patient. Thus, the gradual equilibrium of Admati and Perry appears not to be robust to asymmetries in agents' valuations.

Our paper considers an environment similar to that in Compte and Jehiel (2003), but with multiple agents of each type, and no pre-specified order of play. We show that cycles emerge endogenously in equilibrium if the project is large enough. In this equilibrium, agents make partial contributions to the public good, recovering the gradualism in Admati and Perry (1991). Moreover, we show that in the special case in which there is exactly one agent of each type, our model produces the result of Compte and Jehiel (2003) in which the project is completed in two steps . We conclude that gradualism and alternation can appear as a purely strategic phenomenon provided there are in-group free-riding incentives. The emergence of endogenous contribution cycles is understudied, and in particular, the emergence of repeated cycles of endogenous length is, to the best of our knowledge, new to the literature.

The gradualism present in the equilibrium of our model is captured in the literature through different incentive structures. Compte and Jehiel (2004) consider a bargaining variant of the model in Admati and Perry (1991) in which each party's contribution is interpreted as a voluntary "concession" to their share of surplus. Parties have the option to terminate bargaining, in which case the pay-offs obtained depend on the history of offers or concessions made in the bargaining process. They show that history-dependent outside options can lead to gradualism. Kessing (2007) generalizes the gradualism of Admati and Perry (1991) that arises from convex costs to a differential game counterpart, with $n > 1$ homogeneous agents. Differently to most papers in the literature, Marx and Matthews (2000) focus on history-dependent strategies. They show that allowing contributions to be made slowly over time enhances efficiency in some equilibria, even though individual contributions are

[4] This assumes the cost of provision is not "too high"; otherwise the public good is not provided and agents contribute nothing. This threshold is inefficient, in the sense that some socially desirable projects may not be completed. If the contribution cost is linear, the inefficient equilibrium with gradual contributions still exists, although there is also a no-delay, "one-step completion" equilibrium.

private information. In these equilibria, players' contributions are supported by the threat of stopping cooperation. Finally, Bonatti and Hörner (2011) consider an "experimentation" model of public good provision. Completing the project requires a breakthrough, which in turn requires effort. Some projects can never be completed, and achieving a breakthrough is the only way to ascertain the project's type. Agents' choice of effort is unobserved by the other agents, which leads them to postpone effort, as in other work reviewed here. However, unlike these related papers (including our own), the effort expended decreases over time due to growing pessimism.

3 The Model

We consider a model in which n agents collaborate to produce a public good, which requires q units of investment. There is an infinite horizon, $t = 1, 2, \ldots$. In each period t, an agent $i(t)$ selected at random has the opportunity to invest $e \geq 0$ towards the public good. Contributing e units has a cost $C(e) = ce$, for $c > 0$.[5] We let $e^i(t)$ denote the contribution made by agent i in period t ($e^i(t) = 0$ for all $i \neq i(t)$) and $e(t)$ denote the contribution made in period t by the agent selected in that period. The amount of effort outstanding for completion of the good at the end of period t is the *state*

$$s_t = \max\{s_{t-1} - e(t), 0\}, \quad \text{with } s_0 = q.$$

There are two groups of agents, $m = 1, 2$, and $n^m > 1$ agents of type $m = 1, 2$.[6] For convenience, we refer to group 1 agents as *juniors*, and to group 2 agents as *seniors*. We assume that the probability that any particular individual of group m is selected is $\pi^m = \tilde{\pi}^m / n^m$, where $\tilde{\pi}^1 + \tilde{\pi}^2 = 1$. The payoff of a group-$m$ agent i is:

$$U^i = (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} \left[u^m(s_t) - ce^i(t) \right],$$

where $u^m(s) = 0$ for $s > 0$ and $u^m(0) = \theta^m > 0$.

A time-t *history* $h(t)$ is the tuple $(\{i(\tau)\}_{\tau=1}^{t-1}, \{e(\tau)\}_{\tau=1}^{t-1})$, which encodes previous game play,

[5] With a convex cost, gradualism may be optimal, inducing cycles for technological reasons and not for strategic reasons. While a complete understanding of the interaction of technological constraints and strategic incentives is interesting, we restrict attention to linear costs to isolate strategic incentives.

[6] We cover the instances in which there is at least one group $j \in \{1, 2\}$ with a single member as special cases.

along with the state $s = q - \sum_{\tau=1}^{t-1} e(\tau)$. We say $s^{h(t)}$ is the *corresponding state* for $h(t)$. Let H^t be the set of all time-t histories, and $H := \cup_{t \in \mathbb{N}} H^t$ be the set of all histories in the game. An *equilibrium* is a (group) symmetric subgame perfect equilibrium in pure strategies. A pure strategy for a group-m agent i is a function $\sigma^m(\cdot) : H \to \mathbb{R}_+$, where $\sigma^m(h)$ denotes the contribution of an agent of group-m from history h when she has the opportunity to contribute.

Consider any history h and let h_e be the history induced by group-m agent i contributing $e \in [0, s^h]$ at h. The payoff to i from contributing e when she has an opportunity to play is

$$W^m(e, h) = \delta V^m(h_e) + (1-\delta)\left[u^m(s^h - e) - ce\right], \tag{3.1}$$

where the value of a player i of type m at history h, $V^m(h)$, satisfies *(i)* $V^m(h) = \theta^m$ for any $h \in H$ for which $s^h = 0$, and *(ii)* for any h with $s^h > 0$, letting $\sigma^m(h) = \arg\max_{e \geq 0} W^m(e, h)$,

$$V^m(h) = \sum_{j=1}^{2} \tilde{\pi}^j \left[\delta V^m(h_{\sigma^j(h)}) + (1-\delta) u^m(s^h - \sigma^j(h))\right] - \pi^m c(1-\delta)\sigma^m(h). \tag{3.2}$$

We formally define our notion of equilibrium discussed above, as follows.

Definition 3.1. *A (group) symmetric subgame perfect equilibrium in pure strategies is a pair $(\sigma^1(\cdot), \sigma^2(\cdot))$ such that $\sigma^m(h) = \arg\max_{e \geq 0} W^m(e, h)$ for all $m = 1, 2$ and all histories h with $s^h \leq q$.*

4 Results

In this section, we present our results. We begin by analyzing "small" projects in Section 4.1. We define a project as small if it starts at a history h such that there exists an equilibrium in which the project is completed immediately with positive probability, and denote the size of the largest small project by ϕ_1. We show that in equilibrium, all relevant information about the history is captured by the state, which uniquely pins down agents' strategies. After characterizing equilibria of small projects, we turn to our main analysis in Section 4.2. We show that any project of size $q > \phi_1$ can be seen as a sequence of small projects. Thus, the equilibrium can be characterized recursively. Using these tools, we characterize equilibrium outcomes in large projects, in particular contribution cycles and efficiency.

4.1 Small Projects

We begin by establishing that in equilibrium, there exists a unique threshold $w(1) > 0$ such that the project is completed without delay if and only if $s^h \leq w(1)$. We also show that equilibrium strategies from any history with corresponding state $s \leq w(1)$ are *history-independent*, and uniquely determined by the corresponding state. As will soon become clear, the state is the relevant information for players at *any* history of the game.

Lemma 4.1. *In equilibrium, $\sigma^m(h) = s^h$ for all $m \in \{1, 2\}$ if and only if $s^h \leq w(1)$, where*

$$w^m(1) \equiv \left(\frac{1}{1 - \delta\pi^m}\right)\frac{\theta^m}{c} \quad \textit{and} \quad w(1) \equiv \min\{w^1(1), w^2(1)\}. \tag{4.1}$$

The logic for the result is straightforward. First, given any history h, when the corresponding state s^h is sufficiently small, it is a dominant strategy for any agent to finish the project outright.[7] We next find the threshold $w^m(1)$ such that for all h with $s^h \leq w^m(1)$, it is a best response for a group-m agent to finish the project outright whenever all other agents do the same. By construction, it is an equilibrium for all agents to finish the project outright whenever $s^h \leq w(1) = \min\{w^1(1), w^2(1)\}$. Moreover, since contributions are strategic substitutes, outright completion from others maximizes free riding incentives for any agent. It follows that this is in fact the unique equilibrium for $s^h \leq w(1)$. Finally, note that if $w^m(1) < w^{m'}(1)$, then agents of group m do not complete the project outright whenever they have an opportunity to play at $s^h \in (w(1), w^{m'}(1)]$. Thus from any history h such that $s^h > w(1)$, there is delay with positive probability.

Given the significance of $w^m(1)$ for the construction of equilibrium that follows, we consider the generic case (true for all but a measure zero of the parameter-values) where $w^1(1) \neq w^2(1)$. For presentation purposes we assume that juniors have a higher willingness to finish the project than seniors; i.e., $w^2(1) < w^1(1)$. Since we will show that the equilibrium is Markovian, in the main text we write "from state s" or "at s" to mean "from any history with corresponding state s." We also slightly abuse notation to write $\sigma^m(s)$ as the strategy played by, and $V^m(s)$ as the value to, any group-m agent from any h with $s^h = s$.

As we have seen, seniors do not immediately finish the project when they have an opportunity to play from any $s > w(1)$. Because contributions are strategic substitutes, this implies there is $s' > w^1(1)$ such that any junior agent still completes the project immediately whenever

[7] This is intuitive: if the agent does not finish outright, she can at best expect another agent to finish the project tomorrow for sure. Thus, the payoff for waiting is at most $\delta\theta^m$. On the other hand, finishing outright gives a payoff $\theta^m - c(1-\delta)s^h$ which is higher than $\delta\theta^m$ whenever $cs^h < \min\{\theta^1, \theta^2\}$.

she is recognized at $s \leq s'$. Let ϕ_1 denote the largest such s; that is, the largest state s from which juniors finish the project outright in equilibrium. How would seniors react at states in $(w(1), \phi_1]$? We show that for all s below some threshold $\bar{s}_1 \leq \phi_1$, seniors use a *cutpoint strategy* in equilibrium, gradually moving the project towards completion when they have an opportunity to play.

Definition 4.2. *We say that σ^m is a cutpoint strategy in $[a, b] \subseteq S$ if there is a strictly increasing sequence $\{s(k)\}_{k=0}^{K}$, with K possibly infinite, such that*

1. $s(0) \equiv a$, *and* $\lim_{k \to K} s(k) = b$,
2. $\sigma^m(s) = s - s(k-1)$, $\forall s \in (s(k-1), s(k)]$ *and* $\forall k : 1 \leq k \leq K$.

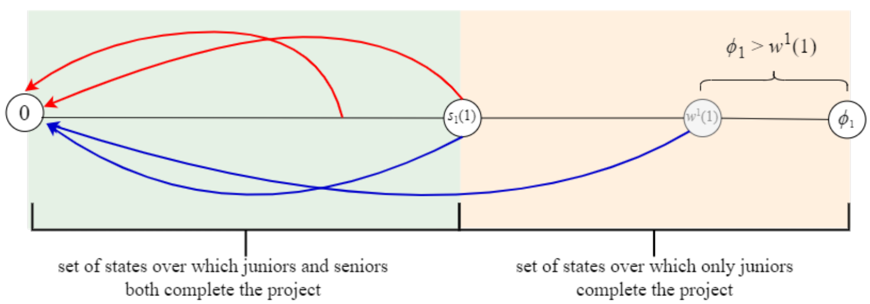


Figure 2: Summary of equilibrium behavior by both sub-groups, close to project completion.

By our previous arguments, in equilibrium seniors follow a cutpoint strategy in $[0, w(1)]$. So let $s_1(0) = 0$ and $s_1(1) = w(1)$, and consider a history h with corresponding state $s > w(1)$ close to $w(1)$. We know already that seniors don't finish the project outright whenever $s > w(1)$. In fact it is easy to see that seniors will contribute at most $s - w(1)$. Since in equilibrium all agents complete the project outright from any state $s' \leq w(1)$, a contribution that moves the project below $w(1)$ without finishing is wasteful. On the other hand, contributing an amount $e \in (0, s - w(1))$ is also wasteful; given that $\sigma^2(s) \leq s - w(1)$, for s close to $w(1)$, seniors are strictly better off by doing nothing. Thus, in equilibrium, from s close to $s(1)$, seniors must either move the project to $s(1)$ or remain inactive.

By an analogous argument to that used to establish the existence of $w(1)$, there must exist s sufficiently close to $w(1)$ such that from s, a senior member is strictly better off expending the effort to move the project to $w(1)$ rather than not contributing. This inequality is preserved up to some point $s_1(2) > w(1)$, at which senior members would be indifferent between contributing $e = s_1(2) - w(1)$ and contributing zero, given the equilibrium conjecture that

$\sigma^2(s) = s_1(2) - w(1)$. Hence, in equilibrium, seniors follow a cutpoint strategy in $[0, s_1(2)]$. We extend this logic recursively for states $s > s_1(2)$, provided at each step k, $s_1(k) < \phi_1$.

The previous argument — which we make precise in the proof of Proposition 4.3 — shows that in equilibrium, seniors follow a cutpoint strategy for all $s \leq \min\{\phi_1, \lim_{k\to K} s_1(k)\}$. This is in line with the logic in Admati and Perry (1991), Compte and Jehiel (2003), and other papers in the literature. Remarkably, though, we show that in this context, in equilibrium this process *must come to a halt at a point* $\overline{s}_1$ *strictly below* ϕ_1. Thus, there is a region $[\overline{s}_1, \phi_1]$ within which seniors do not contribute at all, while juniors finish the project outright (see Figure 3). This is an extreme form of free riding from seniors, who in this region of the state space fully rely on juniors finishing the project right away.

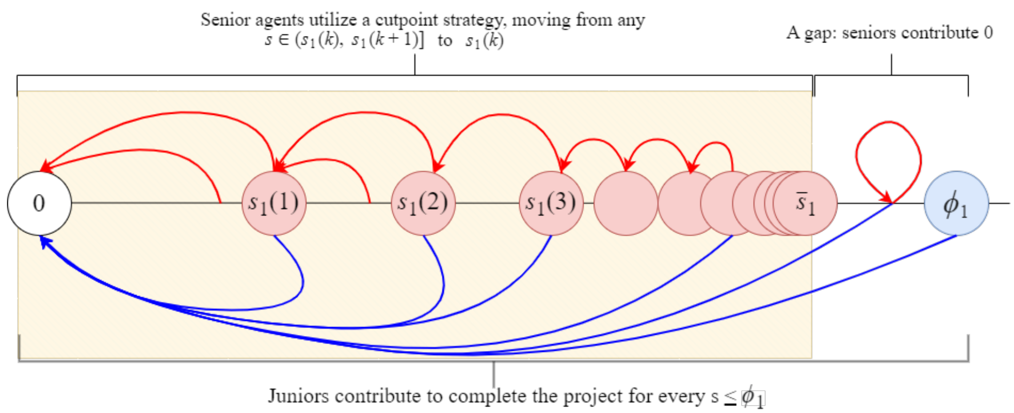


Figure 3: Equilibrium in Small Projects, in the generic case $w^1(1) > w^2(1)$.

Proposition 4.3. *In equilibrium,*

1. *Juniors finish the project outright for all* $s \leq \phi_1$*; i.e.,* $\sigma^1(s) = s$ *for all* $s \leq \phi_1$*, where*

$$\phi_1 = \left(\frac{1}{1 - \delta + \delta\tilde{\pi}^1\left(\frac{n^1-1}{n^1}\right)}\right)\frac{\theta^1}{c}. \tag{4.2}$$

2. *Seniors follow a cutpoint strategy in* $[0, \overline{s}_1] \subset [0, \phi_1]$ *with cutpoints given by*

$$s_1(k) = \frac{1 - \left(\delta\tilde{\pi}^2\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)\right)^k}{1-\delta\tilde{\pi}^2}\frac{\theta^2}{c} < \overline{s}_1 \equiv \left(\frac{1}{1-\delta\tilde{\pi}^2}\right)\frac{\theta^2}{c} \quad \forall k \geq 1, \tag{4.3}$$

3. $\overline{s}_1 < \phi_1$*; seniors do not contribute* ($\sigma^2(s) = 0$) *for all* $s \in [\overline{s}_1, \phi_1]$.

There are three key take-away points from Proposition 4.3. First, small projects have a unique equilibrium. Second, in equilibrium, agents of group $\tilde{m} = \arg\min_m\{w^m(1)\}$ (i.e., juniors) finish the project outright whenever they have an opportunity to move, while agents of the other group (i.e., seniors) make gradual contributions, or no contributions at all. Third, the size of seniors' gradual contributions vanishes as we move farther away from completion, and eventually seniors do not contribute at all. Thus in $[\bar{s}_1, \phi_1]$, only juniors contribute on the path of play. Given this, the values at the "beginning" of the small project are independent of the details of seniors' cutpoint strategy, and given by

$$V^1(\phi_1) = \left(\frac{(1-1/n^1)\tilde{\pi}^1}{1-\delta+(1-1/n_1)\delta\tilde{\pi}^1}\right)\theta^1 \quad \text{and} \quad V^2(\phi_1) = \left(\frac{1-\tilde{\pi}^2}{1-\delta\tilde{\pi}^2}\right)\theta^2. \tag{4.4}$$

To understand part two of the result, note that seniors' behavior drastically changes around each $s_1(k)$. In fact, when a senior has an opportunity to contribute at $s = s_1(k)+\epsilon$, it must be that she prefers to contribute in order to reach the cutoff $s_1(k)$ instead of overshooting and contribute to reach any other $s < s_1(k)$. On the other hand, at $s = s_1(k) - \epsilon$, it must be that she prefers to contribute in order to reach the cutoff $s_1(k-1)$ instead of staying in state $s_1(k) - \epsilon$. Taking the limit as $\epsilon \to 0$ we get

$$\delta V^2(s_1(k)) + c(1-\delta)s_1(k) = \delta V^2(s_1(k-1)) + c(1-\delta)s_1(k-1),$$

which gives

$$\delta V^2(s_1(k)) = \theta^2 - c(1-\delta)s_1(k). \tag{4.5}$$

On the other hand, note that for all k in the sequence, and all $s \in (s_1(k-1), s_1(k)]$, the value (3.2) for a group-m agent is

$$V^m(s) = \tilde{\pi}^1\theta^m + \tilde{\pi}^2\delta V^m(s_1(k-1)) - \pi^m c(1-\delta)\sigma^m(s) \tag{3.2b}$$

Combining (4.5) with (3.2b) for $m = 2$ gives

$$s_1(k) = \frac{1}{1-\delta\pi^2}\frac{\theta^2}{c} + \frac{\delta\pi^2}{1-\delta\pi^2}(n^2-1)s_1(k-1). \tag{4.6}$$

Three points are noteworthy. First, note that the equilibrium cutpoints $s_1(k)$ are uniquely determined. Second, note that solving this difference equation allows us to characterize the equilibrium values $V^m(s)$ for both types, for all $s \le \min\{\phi_1, \lim_{k\to K} s(k)\}$, using equation

(3.2). Third, note that

$$s_1(k) - s_1(k-1) = \frac{1}{1-\delta\pi^2}\left(\frac{\theta^2}{c} - (1-\delta\tilde{\pi}^2)s_1(k-1)\right)$$

is decreasing in k. Thus, seniors' contributions become smaller the farther away from completion the project is. This is one form of delay that arises in equilibrium: seniors partially free ride on both sub-groups' contributions. The further away from completion, the lower the expected value of contributing just a small amount, and the higher the incentives to free ride. To compensate for this, it must be that the cost borne by a senior decreases, which means reducing the amount of the marginal contribution.

To understand the inaction gap ($\bar{s}_1 < \phi_1$), let $\hat{k} \in \{1, 2, \ldots, \infty\}$ be the number of senior cut-points below ϕ_1, the maximal contribution a junior is willing to make to complete the project. As $\hat{k}$ increases, the incentives for juniors to complete the project are compounded, as seniors take progressively smaller steps. A junior thus becomes *more* willing to contribute to avoid even more periods of delay, in expectation, were she not to. That is, ϕ_1 is an increasing function of $\hat{k}$. We show that for each finite $\hat{k}$, $s_1(\hat{k}+1) < \phi_1(\hat{k})$, which rules out any equilibrium with finitely many cutpoints below ϕ_1.

The economic intuition for this result lies in the asymmetry in contribution strategies between the two groups in equilibrium: while seniors remove only a small amount of delay by contributing to their next cutpoint $s_1(\hat{k})$, a contribution from juniors completes the project and thus removes all delay. Because the marginal value of a contribution relative to shirking is thus far higher for juniors than seniors, their willingness to increase their contribution as $\hat{k}$ increases is thus also larger than the contribution seniors are willing to make. In particular, as shown in Figure 4, the movement from $s_1(\hat{k})$ to $s_1(\hat{k}+1)$ is always smaller than the movement from $\phi_1(\hat{k}-1)$ to $\phi_1(\hat{k})$.

In-group and Out-group Effects. At the core of these results is the tradeoff between delay and free-riding for each agent. To understand how this tradeoff is resolved in equilibrium, it is useful to decompose free-riding incentives into two distinct components: *in-group* and *out-group* effects. Agents in our model can free ride on their group peers, and on the other sub-group. The incentive to free-ride on peers is the in-group effect, and is parametrized by n^m. The incentive to free-ride on members of the other sub-group is the out-group effect, and is parameterized by $\tilde{\pi}^m$.

Consider first in-group effects. In principle, in-group effects can have both direct and indirect

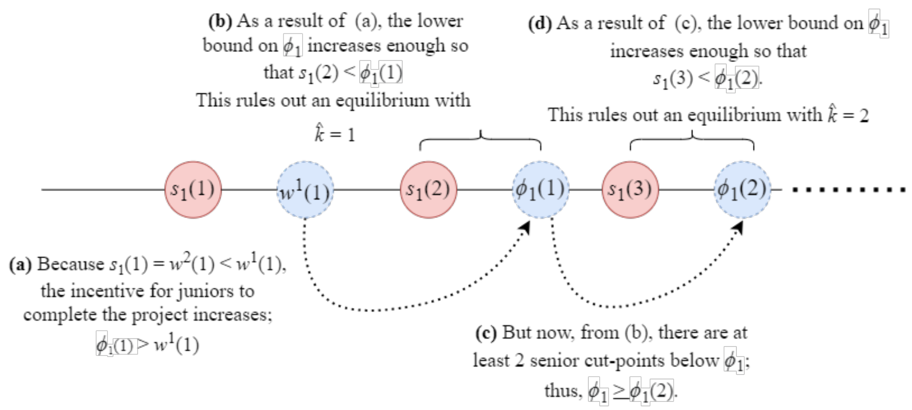


Figure 4: Eliminating the possibility of Finitely Many Cutpoints Below ϕ_1.

consequences for equilibrium. Thus, a change in the number of juniors can affect seniors' equilibrium strategy and vice-versa. However, provided the change in the number of agents in each group does not affect the ranking $w^1(1) > w^2(1)$, in-group effects only affect the behavior of agents of the same group; e.g., the number of junior members affects the size of the small project, ϕ_1, but not the sequence of cutpoints $\{s_1(k)\}$ in seniors' strategy, or its limit $\overline{s}_1$. Similarly, for any k finite, the size of the senior group affects the cutpoint $s_1(k)$, but not the inaction gap or the size of the small project— $\overline{s}_1$ or ϕ_1.

With this noted, characterizing in-group effects in small projects is straightforward. Reducing the number of juniors (reducing in-group free riding incentives) increases the maximum size of the project that juniors are willing to complete outright, ϕ_1. As the numbers of juniors in the team decreases, there is a higher chance that any single junior member would have to contribute in the next period, were she to shirk in the current period. Since at ϕ_1 a junior must be indifferent in equilibrium between shirking and finishing outright, the lower benefit of shirking must have the effect of increasing the amount of effort required to finish the project. As we can see from (4.4), both the direct and the indirect equilibrium effects of reducing the number of juniors in the team have a negative impact on juniors' value at the cutpoint, lowering $V^1(\phi_1)$. For exactly the same reason, reducing the number of seniors in the team increases the size of each increment $s_1(k) - s_1(k-1)$ along the sequence of seniors' strategy cutpoints (see equation (4.6)). However, because ϕ_1 is not affected by the size of the senior group, both $V^1(\phi_1)$ and $V^2(\phi_1)$ are unchanged.

Next, we consider out-group effects. Possibly counterintuitively at first, increasing $\tilde{\pi}_1$ (nominally reducing out-group free riding incentives for juniors) shrinks the size of the largest project that juniors are willing to complete outright, ϕ_1. To understand why this is the case, note that since juniors are the only ones contributing at ϕ_1, the beneficial effect of free-riding to *out-group* agents is quashed. Indeed, in this case, there is no action by seniors to free-ride *on*. Quite to the contrary, each period a senior agent is given an opportunity to contribute, progress on the project remains stalled. Thus, as $\tilde{\pi}^1$ increases, *delay* decreases, lowering the cost of shirking. Hence, ϕ_1 must decrease in order to restore indifference for a junior at the cutpoint. As a result, increasing $\tilde{\pi}_1$ increases $V^1(\phi_1)$ *and* $V^2(\phi_1)$.[8]

So far, we have assumed that $n^m > 1$ for all $m \in M$. Before moving on to efficiency considerations, we consider the special case of single-agent teams.

Single-Agent Teams. The basic structure of equilibria remains essentially unchanged if there is only one agent in any group. However, equilibrium outcomes have some noteworthy features, resulting from the elimination of in-group free riding. The case $n^1 = 1 < n^2$ is unremarkable, with the exception that having a single junior member maximizes the set of projects that are completed immediately with positive probability. Consider next $n^2 = 1 < n^1$. Note that by (4.3), $s_1(k) = w(1) = \overline{s}_1$ for all $k \in \mathbb{N}$. In this case, the single senior agent takes one step whenever $s \leq \overline{s}_1$, and remains inactive in $[\overline{s}_1, \phi_1]$. Since $\overline{s}_1$ and ϕ_1 are not a function of n^2, making $n^2 = 1$ has an unambiguous effect of reducing delay in small projects. Not surprisingly, when $n^2 = 1 = n^1$ we have a combination of both cases: more projects are completed immediately with positive probability, and there is less delay in states close to completion.

Efficiency. In Proposition 4.3 we showed that for any small project with $q \in (w(1), \phi_1]$, there is delay in equilibrium with positive probability. Efficiency, on the other hand, requires that the project be completed if and only if total benefits exceed total costs,

$$\sum_m n^m \theta^m \geq c(1-\delta)q \Leftrightarrow q \leq \frac{n^1\theta^1 + n^2\theta^2}{c(1-\delta)},$$

and that whenever the project is completed, it is completed outright. Since $w(1) = \left(\frac{1}{1-\delta\pi^2}\right)\frac{\theta^2}{c}$, it follows immediately that there is a nonempty set of projects for which the equilibrium is inefficient. In fact, given our equilibrium characterization, we can compute the expected

[8]Reducing $\tilde{\pi}_1$ has the effect of increasing all senior members' cutpoints $\{s_1(k)\}$, as expected. Because the prospect of free riding on juniors is less likely, shirking becomes less attractive. Since at any cutpoint $s_1(k)$, seniors must be indifferent between shirking or moving to the next cutpoint, $s_1(k-1)$, the cost of contributing must increase, increasing $s_1(k)$. Since this holds for all previous cutpoints in the sequence, the effect of reducing $\tilde{\pi}^1$ compounds over time shifting $s_1(k)$ more the higher k is. Thus, in equilibrium, $\partial s_1(k)/\partial\tilde{\pi}^2 > \partial s_1(k-1)/\partial\tilde{\pi}^2 > 0$ for all $k \geq 1$.

time for completion of the project from any state $s \le \phi_1$, as

$$\mathcal{E}(s) = \begin{cases} \frac{1-(\tilde{\pi}^2)^k}{\tilde{\pi}^1} & \text{if } s \in (s_1(k-1), s_1(k)], \text{ for } k = 1, \ldots \\ \frac{1}{\tilde{\pi}^1} & \text{if } s \in [\overline{s}_1, \phi_1]. \end{cases} \tag{4.7}$$

Using (4.3) to substitute the cutpoints in terms of primitives, it follows that for any project $q < \overline{s}_1$,

$$\mathcal{E}(q) = (1 - (\tilde{\pi}^2)^{\psi(q)})/\tilde{\pi}^1, \quad \text{where} \quad \psi(q) \equiv \left\lceil \frac{\log\left(1 - \frac{cq}{\theta^2}(1 - \delta\tilde{\pi}^2)\right)}{\log\left(\delta\tilde{\pi}^2\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)\right)} \right\rceil. \tag{4.8}$$

Note that $\psi(q)$, and therefore $\mathcal{E}(q)$, are weakly increasing in the total cost of the project cq, and weakly decreasing in the value that senior agents put on completion of the project, θ^2. Moreover, as we discussed before, increasing the number of senior agents reduces the size of each increment $s_1(k) - s_1(k-1)$ along the sequence of seniors' strategy cutpoints, and does not affect $\overline{s}_1$ or ϕ_1. As a result, the expected delay for $q < \overline{s}_1$ is also weakly increasing in the number of senior agents in the group. Interestingly, expression (4.8) shows that there is an upper bound on expected delay for any $q \le \phi_1$, given by $1/\tilde{\pi}^1$. Thus, even when a change of parameters leads to a large increase in the k such that $q \in (s_1(k-1), s_1(k)]$, this does not lead to a proportional increase in the expected time for completion.

4.2 Large Projects

We now move to studying the equilibrium of the contribution game for any possible project length q. Our first result justifies the attention we paid to understanding small projects. We show that any project of size $q > \phi_1$ can be seen as a sequence of small projects. Thus, the equilibrium can be characterized recursively, following the analysis of Section 4.1.

Theorem 4.4. *There exists an almost everywhere unique equilibrium in pure strategies. The equilibrium is characterized by an increasing sequence $\{\phi_\tau\}_{\tau=0}^{\overline{\tau}}$, with $\phi_0 = 0$ and $\overline{\tau}$ possibly infinite, such that:*

(i) For any $\tau \ge 1$, there is a unique $m_\tau \in \{1, 2\}$ such that

(a) $\sigma^{m_\tau}(s) = s - \phi_{\tau-1}$ for all $s \in (\phi_{\tau-1}, \phi_\tau]$, and

(b) σ^{-m_τ} is a cutpoint strategy in $[\phi_{\tau-1}, \overline{s}_\tau]$ for a $\overline{s}_\tau < \phi_\tau$, and $\sigma^{-m_\tau} = 0$ in $(\overline{s}_\tau, \phi_\tau]$.

(ii) The sequence $\{\phi_\tau\}$ converges to a limit $\tilde{\phi} > 0$, and for $\tau > 1$,

$$\phi_\tau - \phi_{\tau-1} = \left(\frac{1}{1-\delta+\delta\pi^{m_\tau}(n^{m_\tau}-1)}\right)\frac{\delta V^{m_\tau}(\phi_{\tau-1})}{c}. \tag{4.9}$$

(iii) If $q < \tilde{\phi}$, the project is finished in finite time, and otherwise, the project is never started; i.e., $\sigma^m(s) = 0$ $\forall s \in [\tilde{\phi}, q]$ and $m \in \{1, 2\}$.

The basic intuition for this result relies on two key facts. First, note that at any $s > \phi_1$, agents of both groups are not willing to contribute more than the needed amount to reach ϕ_1, i.e. $\sigma^m(s) \leq s - \phi_1$. It follows that no agent moves the project beyond ϕ_1 at any $s > \phi_1$ (see Lemma A.1 in the Appendix). This implies that for the project to be completed from $s > \phi_1$, the project at some point reaches ϕ_1 and has to transit through ϕ_1 on the equilibrium path. It follows then that the game can be truncated at

$$\phi_1 = \left(1 + \frac{\delta(1-\tilde{\pi}^1)}{1-\delta\pi^1-\delta(1-\tilde{\pi}^1)}\right)w^1(1)$$

by considering ϕ_1 as the terminal state, which pays out $\delta V^m(\phi_1)$ when ϕ_1 is attained.

Second, recall that the equilibrium characterization in Proposition 4.3 relied solely on the labeling assumption that $w^2(1) < w^1(1)$. In a similar fashion, when considering the truncated game at ϕ_1, we define

$$w^m(2) \equiv \frac{1}{1-\delta\pi^m}\frac{\delta V^m(\phi_1)}{c} = w^m(1)\frac{\delta V^m(\phi_1)}{\theta^m}$$

Proposition 4.3 gives that $m_2 = \{m = 1, 2 : w^m(2) > w^{-m}(2)\}$ is the active contributor in this new small project. Equilibrium behavior in this second small project is described by replacing juniors (group 1) for group m_2, $w^m(1)$ for $w^m(2)$, and θ^m for $\delta V^m(\phi_1)$ (Figure 5).

Note that generically, $w^1(\tau) \neq w^2(\tau)$ for any $\tau \geq 1$, so we can continue to characterize the equilibrium for the full game, by recursively truncating the game at each state ϕ_τ that determines the upper bound of step τ, for a given $\phi_{\tau-1}$.[9] Continuing this process, we obtain the same structure recursively. Proceeding in this fashion, parts *(i)* and *(ii)* of the theorem follow immediately from our analysis in Section 4.1. For *(iii)*, note that if $\phi_\tau \to \tilde{\phi} < q$, then

[9] Generically projects are finished in a finite number of steps, or contributions never start. In the former, generically $w^m(\tau) \neq w^{m'}(\tau)$, while in the latter this equality cannot be ruled out by standard measure theoretic arguments. Nevertheless, in any equilibrium with $\overline{\tau} \to \infty$, the case in point, contributions never start generically. In that sense, there could be multiple equilibria that sustain all equilibria in which projects are never finished, and hence, $V^m(q) = 0$.

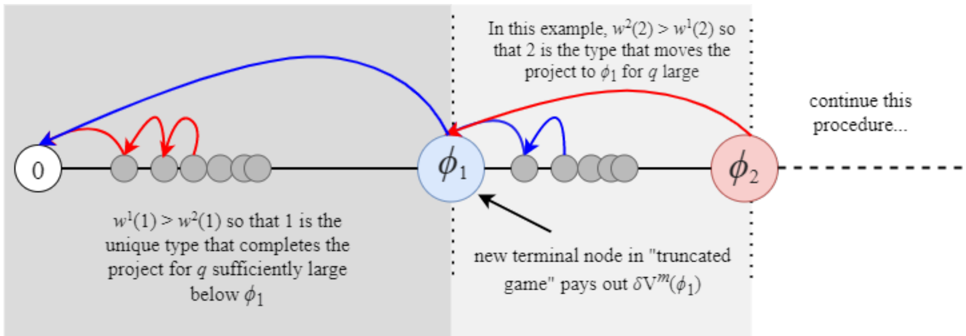


Figure 5: Successively truncating the game at the maximal point at which either type is willing to contribute to the previous terminal node, and applying our equilibrium characterization for small projects, we obtain the equilibrium structure of Theorem 4.4 recursively.

for each $m \in \{1, 2\}$,

$$w^m(\tau) := \frac{\delta V^m(\phi_{\tau-1})}{c(1-\delta\pi^m)} \to 0,$$

for else, $\phi_\tau \geq \phi_{\tau-1} + \min\{w^1(\tau), w^2(\tau)\}$, which would contradict convergence of $\{\phi_\tau\}$. Hence, $V^m(\phi_\tau) \to 0$, so that $V^m(\tilde{\phi}) = 0$. It follows that at any $s > \tilde{\phi}$, contributing nothing is strictly preferred by both types to contributing any positive amount.

4.2.1 Endogenous Contribution Cycles

In this section, we explain the emergence of contribution cycles. From Theorem 4.4, there is generically a unique contributor at each cutpoint ϕ_τ. We call $m(\tau)$ the *active contributor* in step τ, and let $-m(\tau) = \{m \in M : m \neq m(\tau)\}$. From (4.4), and using $\delta V^m(\phi_1)$ as the terminal payoffs in the truncated game, for all $\tau > 1$ we have:

$$V^{m_\tau}(\phi_\tau) = \alpha^{m_\tau} V^{m_\tau}(\phi_{\tau-1}) \quad \text{and} \quad V^{-m_\tau}(\phi_\tau) = \beta^{-m_\tau} V^{-m_\tau}(\phi_{\tau-1}) \tag{4.10}$$

where

$$\alpha^m \equiv \frac{\delta\pi^m(n^m-1)}{\delta\pi^m(n^m-1)+(1-\delta)} \quad \text{and} \quad \beta^m \equiv \frac{\delta(1-\tilde{\pi}^m)}{1-\delta\tilde{\pi}^m}.$$

Clearly, both $\alpha^m < 1$ and $\beta^m < 1$, so agents' values decrease as we move farther away from completion. Moreover, from (4.10),

$$\frac{V^{m_\tau}(\phi_\tau)}{V^{-m_\tau}(\phi_\tau)} = \Delta^{m_\tau} \frac{V^{m_\tau}(\phi_{\tau-1})}{V^{-m_\tau}(\phi_{\tau-1})} \quad \text{where} \quad \Delta^m \equiv \frac{\alpha^m}{\beta^{-m}} < 1, \tag{4.11}$$

reflecting the fact that the value of the active contributor decreases faster than that of the non-active contributor. Note that the equilibrium payoff of non-active contributors only decreases with τ due to impatience and delay, which occurs when one of them is selected to contribute. Thus their value decreases with τ faster the smaller is the probability that a contributor is selected to contribute, $\tilde{\pi}^{m_\tau}$, and the lower is the discount factor δ. On the other hand, the contributor's value decreases with τ due to impatience, delay, *and* due to the cost of moving the project forward. Since this expected cost is smaller the larger the within-group externalities, their value decreases slower the larger n^{m_τ} is.

Since the value of the active contributors decreases with τ faster than the value of the non-active contributors, whenever the same group plays the role of active contributor in successive small projects, initial differences in willingness to pay at any given state ϕ_τ wash out the farther away from ϕ_τ the process is. This opens the possibility that seniors can be active contributors in earlier phases of the game if juniors are exclusively responsible for finishing a sufficiently large number of small projects in the later phases. In particular, note that for any $\tau \geq 1$, an agent of group m is the active contributor for small project τ if and only if

$$\Omega^m(\phi_\tau) \equiv w^m(\phi_\tau)/w^{-m}(\phi_\tau) > 1.$$

Since $w^m(\tau) = \frac{\delta V^m(\phi_{\tau-1})}{c(1-\delta\pi^m)}$, the growth rate of $\Omega^{m_\tau}(\phi_\tau)$ is equal to that of $V^{m_\tau}(\phi_{\tau-1})/V^{-m_\tau}(\phi_{\tau-1})$, and then by (4.11), we have:

$$\Omega^{m_\tau}(\phi_{\tau+1}) = \Delta^{m_\tau}\Omega^{m_\tau}(\phi_\tau). \tag{4.12}$$

Expression (4.12) allows us to determine the identity of the active contributor recursively. Note that by construction we must have $\Omega^{m_\tau}(\phi_\tau) > 1$. Given this, $m_{\tau+1} = m_\tau$ if and only if $\Omega^{m_\tau}(\phi_{\tau+1}) > 1$, or equivalently $\Delta^{m_\tau}\Omega^{m_\tau}(\phi_\tau) > 1$. We use this result to study the emergence of contribution cycles.

We begin by providing a necessary and sufficient condition for agents of different types to alternate in the role of active contributors. Recall that by assumption, $m_1 = 1$ (i.e., junior agents finish the last small project). Suppose juniors remain in the role of active contributors for a stint of $j(1) \geq 1$ consecutive steps. Noting that $\Omega^1(\phi_\tau) = (\Delta^1)^{\tau-1}\Omega^1(\phi_1)$ for all $\tau \leq j(1)+1$, there is a switch in the identity of the active contributor at $j(1)+1$ if and only if

$$\Omega^1(\phi_{j(1)+1}) < 1 < \Omega^1(\phi_{j(1)}) \quad \Leftrightarrow \quad (\Delta^1)^{j(1)} < \Omega^2(\phi_1) < (\Delta^1)^{j(1)-1}.$$

Since $\Delta^1 < 1$, and by assumption $\Omega^2(\phi_1) < 1$, there exists a unique $j(1) \in \mathbb{Z}_+$ that satisfies this inequality, which is given by

$$j(1) = \left\lfloor 1 + \frac{\log w^2(1) - \log w^1(1)}{\log(\Delta^1)} \right\rfloor. \tag{4.13}$$

Together with (4.9), which pins down the size of each small project $\tau \leq j(1)$, (4.13) pins down the maximal project size for which there is no alternation in active contributors. While this *does not* necessarily imply that only juniors contribute on the path of play (as seniors can make partial steps in an initial small project τ if $\phi_{\tau-1} < q < \overline{s}_\tau$), it does rule out contribution cycles for projects that are not sufficiently large, as the project transitions from $\phi_{\tau-1}$ to $\phi_{\tau-2}$, from $\phi_{\tau-2}$ to $\phi_{\tau-3}$, and so on, with contributions by juniors in each small project, with possible delay at each point whenever a senior agent has an opportunity to move.

Remark 4.5. *The equilibrium has no contribution cycles if and only if*

$$q < \left(\frac{1}{1 - \delta\tilde{\pi}^2 - \delta\tilde{\pi}^1/n^1}\right) \frac{1 - (\alpha^1)^{j(1)}}{(1 - \alpha^1)} \frac{\theta^1}{c} = \phi_{j(1)}.$$

Provided $q > \phi_{j(1)}$, the final stint of $j(1)$ small projects in which juniors are active contributors is preceded by a stint of steps in which seniors are active contributors. How does this process evolve for q large? Suppose seniors remain in the role of active contributors for a stint of $j(2) \geq 1$ consecutive steps. Note that for all τ such that $j(1)+1 \leq \tau \leq j(1)+j(2)+1$,

$$\Omega^1(\phi_\tau) = \frac{1}{(\Delta^2)^{\tau-j(1)-1}}\Omega^1(\phi_{j(1)+1}) = \frac{(\Delta^1)^{j(1)}}{(\Delta^2)^{\tau-j(1)-1}}\Omega^1(\phi_1).$$

Thus, there is a switch in the identity of the active contributor at $h(1)+h(2)+1$ if and only if

$$\Omega^1(\phi_{j(1)+j(2)}) < 1 < \Omega^1(\phi_{j(1)+j(2)+1}) \quad \Leftrightarrow \quad \frac{(\Delta^1)^{j(1)}}{(\Delta^2)^{j(2)-1}} < \Omega^2(\phi_1) < \frac{(\Delta^1)^{j(1)}}{(\Delta^2)^{j(2)}}.$$

Since $\Delta^1 < \Omega^2(\phi_1)/(\Delta^1)^{j(1)-1} < 1$ by definition of $j(1)$, there is an $j(2) \geq 1$ that satisfies this inequality. This alternation between juniors and seniors in the role of active contributors continues until the associated sequence of contributions $\phi_\tau - \phi_{\tau-1}$ in each step τ converges to zero, or the cumulative contributions reach q (whichever happens earlier). This gives one of our main results.

Theorem 4.6 (Contribution Cycles)**.** *For any integer $k \geq 1$, define recursively the total* *number of steps up to and including stint k as $J(k) \equiv J(k-1) + j(k)$, letting $J(0) = 0$. Let $\iota(k) = 1$ if k is odd, $\iota(k) = 2$ if k is even, and $\Omega^1(\phi_0) \equiv \frac{1-\delta\pi^2}{1-\delta\pi^1}\frac{\theta^1}{\theta^2}$. Then for all $k \geq 1$:*

$$m_\tau = \begin{cases} 1 & \text{if } J(2k-2) < \tau \leq J(2k-1) \\ 2 & \text{if } J(2k-1) < \tau \leq J(2k), \end{cases}$$

where the number of steps in stint k, $j(k)$, is given recursively by:

$$\Omega^{\iota(k)}(\phi_{J(k)}) = [\Delta^{\iota(k)}]^{j(k)}\Omega^{\iota(k)}(\phi_{J(k-1)}) \qquad \text{and} \qquad \Delta^{\iota(k)} < \Omega^{\iota(k)}(\phi_{J(k)}) < 1. \tag{4.14}$$

Theorem 4.6 shows that in the generically unique equilibrium, agents belonging to distinct sub-groups within the team endogenously alternate actively contributing to the public good, until either the sequence $\{\phi_\tau\}$ converges or q is reached. This defining feature of equilibrium endogenizes the alternation in player contributions that was heretofore exogenously imposed in the related literature, following Admati and Perry (1991). To our knowledge, our paper is the first to demonstrate the emergence of endogenous contribution cycles in equilibrium.

In-group and Out-group Effects. In general, each contribution stint k can consist of multiple small projects. However, in the special case in which $\Delta_1 = \Delta_2 = \Delta$, the ratio of values changes at the same rate when junior and senior agents are active contributors. As a result, asymmetries in the length of the contribution stints by each group lead to imbalances in the ratio of values which are inconsistent with equilibrium. In fact, in this case we must have that with the exception of the final stint, juniors and seniors take turns completing a *single* small project. Now, note that

$$\Delta^m = \frac{(n^m - 1)(1 - \delta\tilde{\pi}^{-m})}{n^m(1 - \delta\tilde{\pi}^{-m}) - \delta\tilde{\pi}^m}$$

Therefore, if groups are symmetric, so that $n^m = n/2$ and $\tilde{\pi}^m = 1/2$ for all $m \in M$, then $\Delta^1 = \Delta^2 = \Delta$. We then obtain the following result:

Corollary 4.7. *If both groups have equal in-group and out-group free-riding incentives, then $j(k) = 1$ for all $k > 1$.*

When $\Delta_1 \neq \Delta_2$, the rate of change of $\Omega^m(\phi_\tau)$ varies with the identity of the active contributor, and one-step cycles may not be sufficient to maintain a balance in the relative willingness to pay. In order to restore equilibrium, then, the length of contribution stints by each group must be asymmetric. In our next result, we characterize the asymmetry in contribution cycles.

Proposition 4.8. *If $\Delta^{m'} < \Delta^m$, contribution cycles alternate between single-step stints by type m' agents and stints of either x^m or $x^m + 1$ small projects by group-m agents, where*

$$x^m \equiv \left\lfloor \frac{\log[\Delta^{m'}]}{\log[\Delta^m]} \right\rfloor$$

□

Two points are worth emphasizing. First, note that the length of each stint $k > 1$ *does not depend on relative preference intensity.* This is because in equilibrium, the last contribution stint by juniors eliminates all differences in willingness to pay across members of both groups due to differences in preference intensity. As a result, in earlier stages of the game, the determination of the identity of the active contributor solely depends on free-riding incentives, as captured by n^m and $\tilde{\pi}^m$.

Second, note that if both groups have equal size, then $\Delta^1 > \Delta^2$ if and only if juniors are more likely to be selected to contribute ($\tilde{\pi}^1 > 1/2$). Similarly, if both groups are equally likely to contribute, then $\Delta^1 > \Delta^2$ if and only if juniors outnumber seniors.

The first case balances out in-group free riding incentives. Proposition 4.8 says that the group of agents who are active contributors in "long" stints of small projects is the group for which out-group free-riding incentives are *smaller*. While this result might appear counterintuitive at first glance, the logic is familiar to us from the analysis of Section 4.1. What determines the length of a contribution stint is how fast the ratio of values $V^{m_\tau}(\phi_{\tau-1})/V^{-m_\tau}(\phi_{\tau-1})$ decreases towards balance when group m is the active contributor. As we showed in Section 4.1, the reason for the counterintuitive sign of the out-group effect on group m agents is that at the cutpoint $\phi(\tau)$, group $-m$ agents don't make positive contributions. Thus, shifting recognition probability to the opposite group effectively does not increase free-riding opportunities for type m agents, and increases delay. As a result, increasing $\tilde{\pi}^{m_\tau}$ leads to a lower depreciation of the value of the active contributors.

The second case balances out-group effects across groups. Proposition 4.8 says that the group of agents who are active contributors in "long" stints of contribution steps is precisely the group for which in-group free-riding incentives are larger. This is because a larger group size means a smaller expected dent in the value of the active contributors in each step, since other agents of the same group can also foot the bill. Thus, it takes a longer stints of small projects to achieve balance in the value ratio.

The previous results fully characterize equilibria of the game. In Section 4.2.2, we analyze the efficiency of equilibrium outcomes. Before doing so, we consider equilibria of the game in the special case in which one or both groups are composed of a single member, so there are no in-group free riding incentives.

Single-Agent Teams. Suppose $n^1 = n^2 = 1$. From our analysis in Section 4.1, the equilibrium for small projects in this case is characterized by cutpoints

$$s_1(1) = \frac{\theta_2/c}{1-\delta\tilde{\pi}_2} \quad \text{and} \quad \phi_1 = \frac{\theta_1/c}{1-\delta}$$

such that $\sigma^1(s) = s \ \forall s \leq \phi_1$, $\sigma^2(s) = s \ \forall s \leq s_1(1)$, and $\sigma^2(s) = 0 \ \forall s \in (s_1(1), \phi_1]$. Since at the cutpoint ϕ_1 the single junior agent is indifferent between completing the project and not contributing, we must have $V^1(\phi_1) = 0$. This implies that for all $s > \phi_1$, only the senior member contributes. It is easy to see then that $\sigma^2(s) = s - \phi_1$ for all $s \in (\phi_1, \phi_2]$ with

$$\phi_2 = \phi_1 + \frac{\delta\tilde{\pi}_1}{1-\delta\tilde{\pi}_2}\frac{\theta_2/c}{1-\delta}$$

and $V^2(\phi_2) = 0$. It follows that the project is completed if and only if

$$q \leq \frac{1}{1-\delta}\left[\frac{\theta_1}{c} + \frac{\delta(1-\tilde{\pi}_2)\theta_2}{c(1-\delta\tilde{\pi}_2)}\right], \tag{4.15}$$

and if completed, is completed in at most two steps.

This result is analogous to the main theorem of Compte and Jehiel (2003). We can think of the above result as extending their finding to the case where recognition among two heterogeneous agents is stochastic instead of sequentially determined. Theorem 4.9 shows that this result depends crucially on there being a single agent of each group, and thus on the non-existence of in-group free-riding incentives.

Next, suppose $n^2 > 1$ and $n^1 = 1$. As in the previous case, here $\phi_1 = (\theta_1/c)/(1-\delta)$ and $V^1(\phi_1) = 0$, and the single type 1 agent is unwilling to contribute for any $s > \phi_1$. The game truncated at final node ϕ_1 now only involves senior agents, who are fully homogeneous, as in Admati and Perry (1991). The equilibrium of the truncated game is characterized by the sequence

$$\phi_\tau = \phi_1 + \frac{\delta\tilde{\pi}^1}{1-\delta\tilde{\pi}^1}\frac{\Delta^2}{(n^2-1)}\left[\frac{1-(\Delta^2)^\tau}{1-\Delta^2} - 1\right]\bar{s}_1 \qquad \text{for } \tau > 1$$

with associated values

$$V^2(\phi_\tau) = (\Delta^2)^{\tau-1}\frac{\tilde{\pi}_1}{1-\delta\tilde{\pi}_2}\theta_2,$$

recovering the structure of the equilibrium in Admati and Perry (1991).

4.2.2 Efficiency Considerations

We now analyze the efficiency of equilibrium outcomes. We consider two distinct questions related to efficiency: (i) whether all welfare-improving projects are completed, and (ii) what is the delay incurred in finishing efficient projects. Finally, we also compare both the completion of welfare-improving projects and delay to those within a team of *homogenous* agents.

Our normalization of utility by factor $1-\delta$ affects both the project's infinite-lasting flow of payoffs and the one-stage cost of each paid contribution. This normalization simplifies the expressions of the value function as θ^m represents both a flow payoff and the discounted utility of the project. For any $\delta < 1$ this double role of θ^m does not affect the arguments above and, in particular, does not affect the efficiency considerations. However, as $\delta \to 1$, any finite contribution in a single period vanishes relative to the discounted utility of the infinite-lasting project. Hence, when discounting disappears, efficiency considerations are more subtle. To simplify the expressions in this section, we consider a project that, once completed, lasts for T finite periods and delivers θ^m in each period. The utility of such a finished project to agent i of group $m = 1, 2$ is

$$U^i = (1-\delta)\sum_{t=1}^{T}\delta^{t-1}\theta^m = (1-\delta)\hat{\theta}^m \quad \text{where} \quad \hat{\theta}^m \equiv \left(\frac{1-\delta^T}{1-\delta}\right)\theta^m.$$ [10]

Implementable Projects. By expression (4.9) in Theorem 4.4, for any $\tau > 1$, if group m is the unique contributor at τ,

$$\phi_\tau - \phi_{\tau-1} = \left(\frac{1}{1-\delta\tilde{\pi}^{-m}-\delta\pi^m}\right)\frac{\delta V^m(\phi_{\tau-1})}{c}.$$

where

$$V^m(\phi_{\tau-1}) = \begin{cases} \alpha^m V^m(\phi_{\tau-2}) & \text{if } m(\tau-1) = m \\ \beta^m V^m(\phi_{\tau-2}) & \text{if } m(\tau-1) \neq m \end{cases}$$

Since $\alpha_m < \beta_m < 1$, the value $V_m(\phi_\tau)$ is decreasing in τ, and goes to zero as $\tau \to \infty$. It follows that the incremental contribution $\phi_\tau - \phi_{\tau-1}$ is decreasing in τ and goes to zero as

[10]Note that $\lim_{\delta\to1}\frac{U^i}{1-\delta} = \lim_{\delta\to1}\hat{\theta}^m = \lim_{\delta\to1}\frac{1-\delta^T}{1-\delta}\theta^m = T\theta^m$. Hence, the payoff of an agent i in group $m = 1, 2$ is $U^i = (1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\left[u^m(s_t) - ce^i(t)\right]$, where $u^m(s_t) = 0$ for $s_t > 0$ and $u^m(0) = \hat{\theta}^m > 0$.

$\tau \to \infty$. This means that the sequence $\{\phi_\tau\}$ converges to a point $\tilde{\phi}$. The threshold $\tilde{\phi}$ gives the size of the smallest project that will not be completed in equilibrium.

Now, note that using (4.9) and the derived values for each agent at the end of any given contribution stint, we can calculate $\phi_{J(k)}$ for any $k \geq 1$. In particular, note that the values at the end of any given stint k only depend on the number of times each agent contributes up to the k^{th} stint. For instance, for $k \geq 1$ odd, we have

$$V^1(\phi_{J(k)}) = [\alpha^1]^{j(k)}[\beta^1]^{j(k-1)}V^1(\phi_{J(k-2)})$$

and

$$V^2(\phi_{J(k)}) = [\beta^2]^{j(k)}[\alpha^2]^{j(k-1)}V^2(\phi_{J(k-2)})$$

Proposition 4.8 then allows us to obtain an upper bound for the limit $\tilde{\phi}$. We present the exact characterization of this bound in the proof of Theorem 4.9. Using this bound, we address the question of whether all projects which would be efficient to pursue – i.e., projects such that $cq < (n^1\hat{\theta}^1 + n^2\hat{\theta}^2)$ – are in fact completed. We call these projects *efficient* for short. We show that not all efficient projects are completed, even in the limit as $\delta \to 1$.

Theorem 4.9. *For any $\delta < 1$, $c\tilde{\phi} < \hat{\theta}^1 + \hat{\theta}^2$. Hence, (i) for any fixed $\delta < 1$, some efficient projects are not completed. Moreover, (ii) since $n^m > 1$ for any $m \in \{1, 2\}$, then even in the limit as $\delta \to 1$, some efficient projects are not completed.*

The qualification that $n^m > 1$ for any $m \in \{1, 2\}$ is important. As we discussed in the previous section, when $n^1 = n^2 = 1$ the project is completed if and only if (4.15) holds, and if completed, it is completed in at most two steps. It follows that

$$c\tilde{\phi}_{n^1=n^2=1} = \hat{\theta}^1 + \frac{1-\tilde{\pi}^2}{1-\delta\tilde{\pi}^2}\delta\hat{\theta}^2$$

which implies that in the limit as $\delta \to 1$, all efficient projects are completed. We thus see that as agents become arbitrarily patient, *in-group* free riding (possible only when $n^m > 1$ for at least one $m \in \{1, 2\}$) is uniquely responsible for inefficiency in project completion.

Delay. As we have shown before, the incremental contribution $\phi_\tau - \phi_{\tau-1}$ in (4.9), is decreasing in τ and goes to zero as the sequence $\{\phi_\tau\}$ approaches the limit $\tilde{\phi}$. It follows that the number of steps required to finish a project of size q, $J(\hat{k}(q))$, where $\hat{k}(q) \equiv \min\{k : \phi_{J(k)} \geq q\}$, (i) is an increasing and convex function of q, and (ii) $J(\hat{k}(q)) \to \infty$ as $q \to \tilde{\phi}$. Because the number of steps in each contribution stint is fixed, this means that the number of contribution stints required to complete a project of size q, $\hat{k}(q)$, is also an increasing and convex function of q, such that $\hat{k}(q) \to \infty$ as $q \to \tilde{\phi}$. In our next result, we use the structure of equilibrium strategies characterized in the previous section to compute the expected delay $\mathcal{E}(q)$ associated with completion of a project of size q. In particular, we first show that the expected completion time of a project that requires $\hat{k}(q)$ contribution stints, $\{j(1), \ldots, j(\hat{k}(q))\}$ is given by

$$\tilde{\mathcal{E}}(q) \equiv \mathcal{E}\left(\{j(\ell)\}_{\ell=1}^{\hat{k}(q)}\right) = \sum_{\ell=1}^{\hat{k}(q)} \left(j(\ell) + \tilde{\pi}_{m_{-\ell}}(\tilde{\pi}_{m_\ell})^{j(\ell)-2}\right),$$

where $\iota(k) = 1$ if k is odd and $\iota(k) = 2$ if k is even.

By Proposition 4.8, moreover, we know that if $\Delta^{m'} < \Delta^m$, contribution cycles alternate between stints of either x^m or $x^m + 1$ steps by group-m agents, and single-step stints by group-m' agents, while if $\Delta^{m'} = \Delta^m$, both groups alternate in single-step stints. We can then split the terms of the above expression in stints taken by each group, and incorporate this information to compute tight bounds on expected delay for a project of size q.

Proposition 4.10. *Suppose $q = \phi_\ell$ for $\ell > j(1)$, and let $\mathcal{E}_1 \equiv j(1) + \tilde{\pi}_2(\tilde{\pi}_1)^{j(1)-2}$. If $\Delta^m \geq \Delta^{m'}$, the expected delay associated with completion of a project of size q is:*

$$\tilde{\mathcal{E}}(q) = \mathcal{E}_1 + \left(\frac{\hat{k}(q) - 1}{2}\right)\left\{(1 + x) + \frac{\tilde{\pi}_m}{\tilde{\pi}_{m'}} + \tilde{\pi}_{m'}(\tilde{\pi}_m)^{x-2}\right\}.$$

for some $x \in [x^m, x^m + 1]$, where $x^m \equiv \left\lfloor \frac{\log[\Delta^{m'}]}{\log[\Delta^m]} \right\rfloor$.[11] *In particular, $\tilde{\mathcal{E}}(\cdot)$ is an increasing and convex function, and $\lim_{q \to \tilde{\phi}} \tilde{\mathcal{E}}(q) = \infty$.*

Proposition 4.10 quantifies delay from any member of the sequence $\{\phi_\ell\}$. Therefore, together with the characterization for delay in small projects given by (4.8), we have a full characterization of delay for arbitrary project length q. Proposition 4.10 connects the two components of efficiency discussed in this paper: delay and – from theorem 4.9 – the result that some efficient projects are not completed, even as $\delta \to 1$. Proposition 4.10 shows that the expected delay increases non-linearly with the size of the project, and goes to infinity as the size of the project approaches the smallest project not completed in equilibrium, $\tilde{\phi}$. Thus, the expected delay is a smooth measure of both types of inefficiency.

[11] We state the result for an odd number of contribution stints. In the proof of Proposition 4.10, we also provide the result for the case of an even number of contribution stints. Note also that the expression of expected delay in the proposition assumes that the project starts exactly at a cutpoint, i.e., $q = \phi_\ell$ for some $\ell > j(1)$. It is straightforward to compute expected delay when the project starts at a point interior to a small project, using the expression (4.8) for expected delay in small projects.

Comparison with Homogenous Teams. Despite the prevalence of inter-group cooperation in the real world, the case of homogenous teams remains an important benchmark in the literature. How does deviating from this benchmark, by introducing groups with distinct incentives, affect equilibrium outcomes?

To help fix ideas, consider $2n$ agents who participate in the completion of a binary project. We compare the performance of a team in which each of the $2n$ agents has an identical value of θ for completing the project, to one in which there are two payoff-distinct groups – n group 1 members obtain payoff $\theta + \varepsilon$ and n group 2 members $\theta - \varepsilon$, for some $\varepsilon \in [0, \theta]$, upon completion. For ease of exposition, we assume that in both teams, each agent is equally likely of being recognized to contribute (i.e., has probability $1/2n$), and each has per-unit contribution cost c. Thus, the set of welfare-enhancing projects is the same across both teams. In line with the previous analysis, for each team, we examine both the set of implementable projects and expected delay, conditional on completion, for any δ and n. We are particularly interested in the case where $\varepsilon \searrow 0$, as this indexes the *marginal effect* of introducing inter-group heterogeneity within a team.

Implementable Projects. We begin by noting that equilibrium behavior within the homogenous set of agents is akin to that of a two-group team in which the recognition probability of the "other group" vanishes. That is, the behavior of any agent within this arrangement mirrors that of seniors in the two-group case, under the proviso that the recognition probability of juniors, $\tilde{\pi}_1$, goes to zero, even as they remain the group with the larger incentive to complete.[12] We denote the cut-points used by the agents $\{t(k)\}_{k=1}^{\infty}$. Using expression (4.3), we obtain

$$t(k) = \frac{1}{1-\delta}\left[1 - \left(\delta \cdot \frac{2n-1}{2n-\delta}\right)^k\right]\frac{\theta}{c} < \bar{t} = \frac{\theta}{c(1-\delta)}, \tag{4.16}$$

where $\bar{t} \equiv \lim_{k\to\infty} t(k)$, and thus represents the largest project implementable by the team of homogenous agents.

On the other hand, our previous analysis directly allows us to obtain the maximum project size $\tilde{\phi}$ undertaken by the two-group team. In particular, our assumptions guarantee that

[12]As in the rest of the paper, here we restrict to symmetric SPE. A possible question is whether we could support the equilibrium behavior of the two-group case in the single group case, by arbitrarily dividing members into two equal size groups and impose that members of group 1 complete the project outright from any cutpoint $s_1(k)$ of group-2 members. One can check that in fact this is a valid equilibrium construction; *it is however not robust.* In particular, when costs deviate even an arbitrarily small degree from linearity to become convex— i.e., $c(e) = c \cdot e^{1+\nu}$ for ν small— an equilibrium of this form no longer exists. On the other hand, the equilibrium we consider for homogenous agents is robust to this perturbation (it is the limiting equilibrium as $\nu \to 0$). The same is true for the equilibrium described for the two-group case throughout the paper, where groups are distinguished by tangible heterogeneity in values or recognition probabilities.

$\Delta^1 = \Delta^2$, so that by Corollary 4.7, $j(k) = 1$ for $k > 1$.[13] Note that the group 1 agents— who have a higher value for the project— will be active near completion. Thus, from Theorem 4.4, the values to members of each group, respectively, at the start of the final small project are given by $V^1(\phi_1) = \alpha/\delta \cdot (\theta + \varepsilon)$ and $V^2(\phi_1) = \beta/\delta \cdot (\theta - \varepsilon)$, where

$$\alpha = \frac{\delta(n-1)}{2n - \delta(n+1)}, \qquad \beta = \frac{\delta}{2-\delta}.$$

Clearly, $m_2 = 2$ if and only if $V^2(\phi_1) > V^1(\phi_1)$. We thus obtain a global one-to-one alternation in group responsibility over small projects, whenever $\varepsilon < \left(\frac{1-\delta}{(2-\delta)n-1}\right)\theta$. Suppose this inequality holds. Using (4.9) and (4.10) recursively, we directly obtain

$$\phi_k = \frac{2n}{c(2n-\delta\,(n+1))}\left\{\left[\sum_{\ell=0}^{\lfloor\frac{k-1}{2}\rfloor}(\alpha\beta)^\ell + \beta\cdot\sum_{\ell=0}^{\lfloor\frac{k}{2}\rfloor-1}(\alpha\beta)^\ell\right]\theta + \left[\sum_{\ell=0}^{\lfloor\frac{k-1}{2}\rfloor}(\alpha\beta)^\ell - \beta\cdot\sum_{\ell=0}^{\lfloor\frac{k}{2}\rfloor-1}(\alpha\beta)^\ell\right]\varepsilon\right\},$$

with

$$\lim_{k\to\infty}\phi_k \equiv \tilde{\phi} = \frac{2n}{c(2n-\delta\,(n+1))}\left(\frac{1}{1-\alpha\beta}\right)\left((1+\beta)\theta + (1-\beta)\varepsilon\right).$$

We see from above that, within the prescribed range for ε, the size of the largest implementable project is *increasing* in inter-group heterogeneity (i.e., in ε).

We now consider the *marginal introduction* of heterogeneity within the team by letting $\varepsilon \to 0$ in the above expression. The largest implementable project in the limit is

$$\tilde{\phi} = R_\delta(n)\cdot\frac{\theta}{c}, \qquad \text{where } R_\delta(n) \equiv \frac{2n}{2n - 2\delta(n+1) + \delta^2},$$

substituting for α and β, and simplifying. By comparison with $\bar{t}$ in (4.16), the gain in the size of the largest implementable project in the two-group setting over the homogenous team is given by

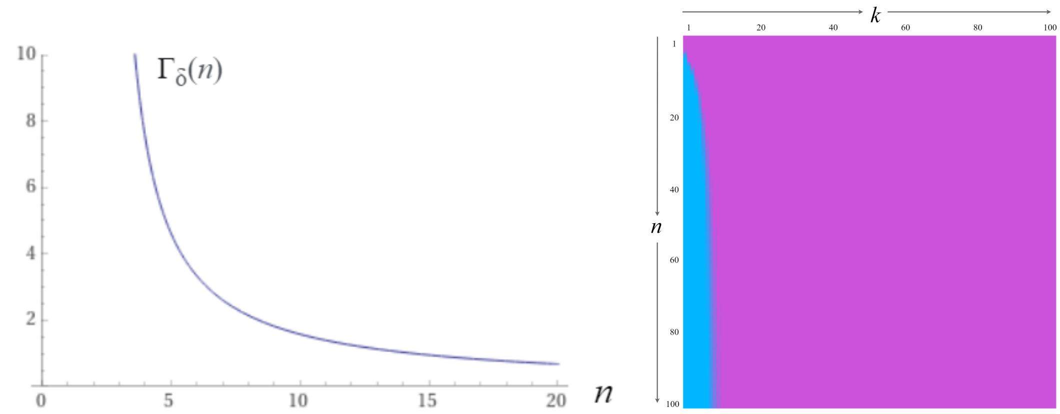
$$\Gamma_\delta(n)\cdot\frac{\theta}{c}, \qquad \text{with } \Gamma_\delta(n) \equiv R_\delta(n) - 1/(1-\delta).$$

It follows that the two-group team is able to complete a larger set of welfare-enhacing projects whenever $\Gamma_\delta(n) > 0$. But, by direct inspection, $\Gamma_\delta(n)$ is a strictly decreasing function of n, with $\lim_{n\to\infty}\Gamma_\delta(n) = 0$. Hence, this is universally the case for *any* δ, n.

We thus illustrate an important principle: *the introduction of within-team heterogeneity in the form of value-differentiated groups improves the size of the largest implementable project*

[13]The incentives to free-ride are the same across both groups, since each has the same recognition probability, and the same number of agents within it.

by an amount bounded away from 0, for any fixed δ, n. As n grows large, this benefit decreases and becomes zero in the limit, as shown in the left panel of Figure 6.



Project Size   Expected Delay

Figure 6: Efficiency Comparison Across Homogenous and Two-Group Teams. The left panel plots the difference in the size of the largest implementable project between two groups and a homogenous team, as a function of team size. The pink (blue) area on the right panel plots the set of (n, k) for which expected delay is smaller (larger) in the two group team than in the homogeneous team.

Delay. We now compare project completion times across the two team structures, focusing again on the case where $\varepsilon \searrow 0$. The analysis we have done so far is nearly sufficient to undertake this comparison. Suppose $q = \phi_k$, as in the characterization of delay given by Proposition 4.10. Consider first the two-group team. We know the expected completion time for a project of size q is exactly $2k$ periods, since each small project takes *two* periods to complete in expectation. This is because exactly one of the two teams completes it outright, and the recognition of a member of either team in each period is $\frac{1}{2}$. On the other hand, the homogenous team moves precisely one cutpoint—that is, from $t(\ell)$ to $t(\ell - 1)$—in each period. Hence, the two-group team achieves project completion strictly faster than the homogenous team if and only if $\phi_k > t(2k)$.

Moreover, because $\lim_{k\to\infty} \phi_k = \tilde{\phi} > \bar{t} = \lim_{k\to\infty} t(2k)$, it follows that for any δ, n, there exists $k^*(\delta, n) \in \mathbb{N}$ such that $\phi_k > t(2k)$ whenever $k \geq k^*(\delta, n)$. In the right panel of Figure 6, the pink region represents those (n, k) for which the two-group team completes the project ϕ_k strictly faster than the homogenous team, while the blue region indicates where the homogenous team performs weakly better. Hence, $k^*(\delta, n)$ is non-trivial— that is, for large n and δ, projects of relatively small size are completed more efficiently with homogenous teams. As the total project size increases, however, eventually the two-group team always dominates in performance.

5 Conclusions

In this paper, we study inter-group collaboration within a team of agents through a model of sequential contributions to a joint project, in a decentralized environment. We assume the joint project is a *binary public good*, delivering benefits inexcludably to the team of agents only once completed, and that there is no deadline or asymmetric information among agents. Crucially, we focus on the case where the team is naturally divided into two sub-groups, each of which has a distinct value from completing the project. This particular problem has several real-world analogues— concerted lobbying amongst firms from different economic sectors, a socio-economically divided population engaging in collective action, and junior and senior scholars collaborating on research, to name but a few.

We show that if the project is sufficiently large, the unique equilibrium of the model displays endogenous contribution cycles, in which agents from different groups within the team alternate making gradual contributions towards project completion. Importantly, *delay*, *inaction*, and *alternation* in work between the sub-groups — familiar features of the teamwork setting— all emerge as features of equilibrium. Neither collaboration between homogenous, nor fully heterogenous agents captures these stylized facts. This points to the salience of the *inter-group* case as a descriptive framework for collaborative settings whenever contributions are transparent and decentralized. Indeed, even when not facially evident, it is natural to think that divisions and self-categorizations within a team certainly do exist. Our results are suggestive of the important role played by such group fragmentation.

By focusing on the case of two sub-groups, we are able to clearly examine the roles of *in-group* and *out-group* free riding, which underpin our results. First, our results show that *delay is a clear consequence of in-group free riding.* With $n^m = 1$, in-group free riding incentives dissipate and the single group-m agent contributes at most once. The existence of multiple agents in each group is thus responsible for delay and gradualism. On the other hand, *out-group free riding is responsible for inaction.* In particular, as the group of agents with a smaller incentive to move the project forward (say seniors) takes more and more steps to complete a progressively larger task, the incentive for juniors to move forward in order to avoid delay on the path of play increases. This leads to an extreme form of out-group free riding wherein any junior with an opportunity to contribute completes the small project with a single contribution, and senior agents remain inactive. Together, gradualism and inaction (or in- and out-group free riding) lead to the emergence of a "small project" as the core segment of work in equilibrium, and the emergence of alternation through endogenous contribution cycles across the two sub-groups.

In the paper, we focused on binary public goods/joint projects. In this case, members of the team only reap benefits when the project is completed. In other cases, agents may *also* obtain a flow payoff even if the project is incomplete; i.e., $u^m(s(t)) = r^m(q - s(t))$ for $s(t) > 0$ and $u^m(0) = qr^m + \theta^m > 0$. All of our analysis extends to this case, with suitable modifications, when the flow payoff r^m is "small." Thus, our conclusions can be extended to applications in which the terminal payoff dominates. When the flow payoff r^m is large relative to terminal payoffs θ^m, however (e.g., collaboration across countries to reduce carbon emissions), the strategic analysis changes qualitatively. This is because in binary public goods settings, agents' efforts are strategic complements across time (in that a higher contribution by one agent induces larger contributions in the future), while in continuous public good settings, efforts are strategic substitutes across time (i.e., contributions lead to a reduction of effort in the future). We believe that extending the analysis in this paper to the continuous public good case would be a promising avenue for future research.

In addition to the theoretical contributions we have emphasized throughout our discussion, our paper also provides guidance for applied research. In the equilibrium of our model, agents with the higher valuation for completion ex ante will assume responsibility for finishing the project in the final stages, but do not necessarily make larger contributions than agents with lower ex ante valuation at all points in time. Moreover, the prevalence of inaction across groups, and the length of contribution stints by agents in both groups are solely determined by in- and out-group free riding incentives, and not ex ante valuations for completion. This suggests that in this context, strategic considerations are a key element to be considered in linking parameters to observed data.

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A Proofs

Proof of Proposition 4.3. We prove this in a series of steps.

Step 1. *Suppose $s_1(2) > w(1)$ is such that for all histories h with $s^h \leq s_1(2)$, $\sigma^1(h) = s^h$. Then:*

1. *For any h with corresponding state $s^h \in (s_1(1), s_1(2)]$, $\sigma^2(h) = s^h - w(1)$.*
2. *For any h with $s^h > s_1(2)$, $\sigma^2(h) \leq s^h - s_1(2)$.*

Proof. Suppose h is such that $s^h \in (s_1(1), s_1(2)]$, and consider the problem of a senior (i.e., group 2) agent, i. We have shown that for any history h' with $s^{h'} > w(1)$, $\sigma^2(h') < s^{h'}$. Moreover, contributing $e \in (s^h - w(1), s^h)$ from h is clearly dominated, since by moving the state to $w(1)$, the project will be finished at the same time, with lower expected costs. Thus, $\sigma^2(h) \in \{0, s^h - w(1)\}$.

Suppose first $\sigma^2(h) = s^h - w(1)$ for all histories h with $s^h \in (s_1(1), s_1(2)]$. Note that since in equilibrium all agents finish the project outright for any history h' with $s^{h'} \leq w(1)$, if agent i contributes $e = s^h - w(1)$, from the history h, she gets a payoff

$$\begin{aligned} W^2(s^h - w(1), h) &= \delta V^2(h_{s^h - w(1)}) - (1-\delta)c(s^h - w(1)) \\ &= \delta\left[\theta^2 - \pi^2 c(1-\delta)w(1)\right] - (1-\delta)c(s^h - w(1)). \end{aligned} \tag{A.1}$$

Now suppose i deviates and contributes $e \in [0, s^h - w(1))$, say $e = s^h - w(1) - \varepsilon$, with $0 < \varepsilon \leq s^h - w(1)$. This gives her a payoff

$$\begin{aligned} W^2(e, h) &= \delta V^2(h_e) - (1-\delta)c(s^h - w(1) - \varepsilon) \\ &= \delta\left[\tilde{\pi}^1\theta^2 + \tilde{\pi}^2\delta V^2(h'') - \pi^2(1-\delta)c\varepsilon\right] - (1-\delta)c(s - w(1) - \varepsilon) \\ &= \delta\theta^2(\tilde{\pi}^1 + \delta\tilde{\pi}^2) - (\delta)^2\tilde{\pi}^2(1-\delta)\pi^2 cw(1) - (1-\delta)c(s - w(1)) + \varepsilon(1-\delta)c(1 - \pi^2\delta), \end{aligned} \tag{A.2}$$

where h'' is the history induced from h_e by a senior contributing to $w(1)$. Note that this expression is increasing in ε. Thus, the binding deviation is $e = 0$, or $\varepsilon = s^h - w(1)$. This is not a profitable deviation iff

$$s^h \leq \left(\frac{\delta\tilde{\pi}^2}{1 - \delta\pi^2}\right)\frac{\theta^2}{c} + \left\{\frac{1 - \delta\pi^2(2 - \delta\tilde{\pi}^2)}{1 - \delta\pi^2}\right\}w(1)$$

$$\Leftrightarrow s^h \leq w(1) + w(1)\left[\frac{\delta\pi^2(n^2 - 1)}{1 - \delta\pi^2}\right] = s_1(2).$$

Suppose instead that in equilibrium $\sigma^2(h) < s^h - w(1)$ for some history h with $s^h \in (s_1(1), s_1(2)]$. Note that the equilibrium payoff for a senior agent at h is bounded above by the RHS of A.2. On the other hand, a deviation to $e = s^h - w(1)$ at h gives a payoff A.1. Since $s^h \in (s_1(1), s_1(2)]$, it follows that this is a profitable deviation, and thus in equilibrium we must have $\sigma^2(h) = s^h - w(1)$ for all histories h with $s^h \in (s_1(1), s_1(2)]$.

For part 2, note that our previous argument shows that $\sigma^2(h) < s^h - w(1)$ for all h with $s^h > s_1(2)$. Now, for any such h, moving the project to any history with corresponding state $s \in (s_1(2), w_1)$ is clearly dominated by moving the project to a history with corresponding state $s_1(2)$, since the project will be finished at the same time, with lower expected costs. Thus $\sigma^2(h) \leq s^h - s_1(2)$ for all $s^h > s_1(2)$.

By definition 2, when senior agents play a cutpoint strategy in $[0, b]$, their strategy is *history-independent* from all histories h for which $s^h \leq b$. As introduced in the main text, we thus use the convention of writing $\sigma^m(s)$ in place of $\sigma^m(h)$, and $V^m(s)$ in place of $V^m(h)$ for any history h such that $s^h = s$, whenever group-m agents play a cutpoint strategy.

Step 2. *Suppose there is $b \in \mathbb{R}_+$ such that senior (group 2) agents follow a cutpoint strategy in $[0, b]$ and $\sigma^1(h) = s^h$ at all h for which $s^h \leq b$. Then*

1. The sequence of cutpoints is described by (4.3):

$$s_1(k) = \frac{1 - \left(\delta\tilde{\pi}^2\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)\right)^k}{1-\delta\tilde{\pi}^2}\frac{\theta^2}{c}$$

2. The values for junior (group 1) agents are given by

$$V^1(s_1(1)) = \theta^1 - \pi_1 n^1 c(1-\delta)s_1(1), \tag{A.3}$$

and for $k > 1$,

$$\begin{aligned} V^1(s_1(k)) = &\left(\frac{1-(\tilde{\pi}^2\delta)^k}{1-\tilde{\pi}^2\delta}\right)\left[\tilde{\pi}^1\theta^1\right] + (\tilde{\pi}^2\delta)^k\theta^1 \\ &- \pi^1\left(\frac{1-\delta}{1-\delta\tilde{\pi}^2}\right)\left[\left(\frac{1-(\delta\tilde{\pi}^2)^k}{1-\delta\tilde{\pi}^2}\right) - \left(\delta\tilde{\pi}^2\right)^k\left(\frac{n^2-\delta\tilde{\pi}^2}{1-\delta\tilde{\pi}^2}\right)\left[1 - \left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)^k\right]\right]\theta^2 \end{aligned} \tag{A.4}$$

Proof. Using (4.5) for $s_1(k)$ and $s_1(k-1)$ in (3.2b) gives

$$\begin{aligned} s_1(k) &= \frac{1}{c}\left(\frac{n^2}{n^2-\delta\tilde{\pi}^2}\right)\theta^2 + \delta\tilde{\pi}^2\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)s_1(k-1) \\ &= w_2(1) + \delta\tilde{\pi}^2\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)s_1(k-1) \end{aligned} \tag{A.5}$$

Solving the difference equation, we have:

$$s_1(k) = \left[\sum_{\ell=0}^{k-1}\left(\delta\tilde{\pi}^2\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)\right)^\ell\right]\left[\frac{1}{c}\left(\frac{n^2}{n^2-\delta\tilde{\pi}^2}\right)\theta^2\right]$$

and using that $\sum_{\ell=0}^{k-1} x^\ell = \frac{1-x^k}{1-x}$ we get (4.3).

The values of junior agents for $k = 1$ follows trivially from the fact that agents in both groups contribute to complete the project at $s_1(1)$. Consider $k > 1$. Note that the value for a junior at $s_1(k)$ is

$$V^1(s_1(k)) = \tilde{\pi}^1\theta^1 + \tilde{\pi}^2\delta V^1(s_1(k-1)) - \pi^1 c(1-\delta)s_1(k) \tag{A.6}$$

and

$$V^1(s_1(k-1)) = \tilde{\pi}^1\theta^1 + \tilde{\pi}^2\delta V^1(s_1(k-2)) - \pi^1 c(1-\delta)s_1(k-1)$$

So substituting, recursively,

$$V^1(s_1(k)) = \left(\frac{1-(\tilde{\pi}^2\delta)^{k-1}}{1-\tilde{\pi}^2\delta}\right)\tilde{\pi}^1\theta^1 + (\tilde{\pi}^2\delta)^{k-1}\theta^1 - \pi^1 c(1-\delta)\left[\sum_{\ell=0}^{k-1}(\tilde{\pi}^2\delta)^{\ell}s_1(k-\ell)\right].$$

Substituting $s_1(k-\ell)$, and simplifying, we have the expression in the statement of step 2.

Step 3. *In any symmetric SPE, σ^2 is a cutpoint strategy in $[0, \overline{s}_1]$ for $\overline{s}_1 \equiv \sup_{k\geq 1} s_1(k)$ if $\sigma^1(h) = s^h$ for all histories h with $s^h \leq \overline{s}_1$.*

Proof. We prove this result by induction. First note that by lemma 4.1 and step 1 above, the statement is true up to $k = 2$, with $s_1(0) \equiv 0$, $s_1(1) \equiv w(1)$, and $s_1(2)$ as defined in step 1. Next, suppose the statement is true up to k, and consider any history h with $s^h \in (s_1(k), s_1(k+1)]$. Note that if we show that for any such h, $\sigma^2(h) \leq s^h - s_1(k)$, an argument analogous to that of step 1 establishes the induction step. To prove this, we show that for any history h with $s^h \in (s_1(k), s_1(k+1)]$, if $\sigma^2(h) > s^h - s_1(k)$, a senior (group 2) agent would gain by deviating to contributing zero.

Note that since moving to a non-cutpoint is dominated by moving to a cutpoint, it is enough to consider $\sigma^2(h) = s^h - s_1(k-\ell)$ for all $\ell : 1 \leq \ell < k$. In the proposed equilibrium, a senior agent that gets to contribute would get a payoff

$$W^2(s^h - s_1(k-\ell), h) = \delta V^2(s_1(k-\ell)) - (1-\delta)c(s - s_1(k-\ell)) = \theta^2 + r^2 q - (1-\delta)cs$$

If instead she deviated to contributing zero, she would get $W^2(0,h) = \delta V^2(h)$. Since by assumption $\sigma^2(h) = s^h - s_1(k-\ell)$ and we know $\sigma^1(h') = s^{h'}$ for all histories with $s^{h'} \leq \overline{s}_1$, at s we have

$$V^2(h) = \tilde{\pi}^2\delta V^2(s_1(k-\ell)) + (1-\tilde{\pi}^2)\theta^2 - \pi^2 c(1-\delta)[s^h - s_1(k-\ell)]$$

Therefore

$$W^2(0,h) = \delta\theta^2 - \delta\tilde{\pi}^2(1-\delta)cs_1(k-\ell) - \delta\pi^2 c(1-\delta)[s^h - s_1(k-\ell)]$$

Thus, contributing zero is a profitable deviation iff

$$\left(\frac{n^2-\delta\tilde{\pi}^2}{n^2}\right)s^h - \delta\tilde{\pi}^2\left(\frac{n^2-1}{n^2}\right)s_1(k-\ell) > \frac{\theta^2}{c}$$

Note that if this is satisfied for $\ell = 1$, it is satisfied for all $\ell : 1 \leq \ell < k$. Moreover, recall that $s^h > s_1(k)$, so writing $s^h = s_1(k) + \varepsilon$, for $\varepsilon > 0$, this is a profitable deviation iff

$$\left(\frac{n^2-\delta\tilde{\pi}^2}{n^2}\right)(s_1(k)+\varepsilon) - \delta\tilde{\pi}^2\left(\frac{n^2-1}{n^2}\right)s_1(k-1) > \frac{\theta^2}{c}$$

and using (A.5), the inequality becomes $\left(\frac{n^2-\delta\tilde{\pi}^2}{n^2}\right)\varepsilon > 0$, which always holds.

Having shown that for any history h with $s^h \in (s_1(k), s_1(k+1)]$, $\sigma^2(h) \leq s^h - s_1(k)$, an argument analogous to that of step 1 shows that in equilibrium $\sigma^2(h) = s^h - s_1(k)$ for any h with $s^h \in (s_1(k), s_1(k+1)]$. This concludes the proof of step 3.

Step 4. *Suppose $\overline{s}_1 < \phi_1$ and $\sigma^1(h) = s^h$ from any h with $s^h \leq \phi_1$. Then $\sigma^2(h) = 0$ from all histories h with $s^h \in [\overline{s}_1, \phi_1]$.*

Proof. (i) First, note that for all h with $s^h \in [\overline{s}_1, \phi_1]$, $\sigma^2(h) < s^h - s_1(k)$ for all k in the sequence. The formal argument is the same the proof of step 3, and therefore omitted here.

(ii) Now suppose for $s^h \in [\overline{s}_1, \phi_1]$, $\sigma^2(h) \in [0, s^h - \overline{s}_1]$, say $\sigma^2(h) = s^h - \overline{s}_1 - \varepsilon$, with $0 \leq \varepsilon \leq s^h - \overline{s}_1$. This gives a group-2 agent payoff

$$W^2(e,h) = \delta V^2(h_e) - (1-\delta)c(s^h - \overline{s}_1 - \varepsilon)$$

Now, by part (i), any contribution $e \in [0, s^h - \overline{s}_1]$ will never result in a move to a state in which (directly or indirectly), a senior (group 2) agent advances the project to any $s_1(k)$ in the sequence. Thus $V^2(h_e) \leq (1-\tilde{\pi}^2)\theta^2$. Hence, for any $e = s - \overline{s}_1 - \varepsilon$, with $0 \leq \varepsilon \leq s - \overline{s}_1$,

$$W^2(e,h) \leq \delta(1-\tilde{\pi}^2)\theta^2 - (1-\delta)c(s^h - \overline{s}_1 - \varepsilon)$$

It follows that if $\sigma^2(h) > 0$ for any such h, a senior (group 2) agent would gain by deviating to contributing zero.

To complete the proof, we show that if $\sigma^2(h) = 0$ for all h with $s^h \in [\overline{s}_1, s_1(1)]$, type 2 agents don't have a profitable deviation in this interval. Note that for any such h:

$$V^2(h) = \left(\frac{1-\tilde{\pi}^2}{1-\delta\tilde{\pi}^2}\right)\theta^2$$

A deviation to any cutpoint $s_1(k)$ gives

$$\begin{aligned}W^2(s^h - s_1(k), h) &= \delta V^2(s_1(k)) - c(1-\delta)(s - s_1(k)) \\ &= \theta^2 - (1-\delta)cs_1(k) - c(1-\delta)(s - s_1(k)) = \theta^2 - (1-\delta)cs\end{aligned}$$

where we used (4.5). This is a profitable deviation iff $W^2(s^h - s_1(k), s) > \delta V^2(h)$ or

$$s^h < \left(\frac{1}{1-\delta\tilde{\pi}^2}\right)\frac{\theta^2}{c} = \overline{s}_1 \Rightarrow\Leftarrow .$$

Step 5. *In any equilibrium, there are infinitely many cupoints $\{s_1(k)\}$ below ϕ_1.*

Proof. Assume, towards a contradiction, that there exists a $\hat{k} < \infty$ such that

$$s_1(\hat{k}) < \phi_1(\hat{k}) \leq s_1(\hat{k}+1).$$

Recall from the text that, by definition, $\phi_1(\hat{k})$ coincides with ϕ_1 when there are exactly $\hat{k}$ cutpoints of juniors below ϕ_1. The above is thus a necessary condition to sustain an equilibrium with exactly $\hat{k}$ senior cutpoints. The above steps establish that group-2 agents play a history-independent strategy from any h with $s^h < \phi_1$; moreover, group-1 agents play a strategy of outright completion from any h with $s^h \leq \phi_1$, we use the notation $\sigma^m(s)$ and $V^m(s)$, again, to mean the strategy and value of group-m agents from any history h with $s^h = s$.

(a) Note that if $s_1(\hat{k}) < \phi_1(\hat{k}) \leq s_1(\hat{k}+1)$, it must be that $\sigma^2(s) = s - s_1(\hat{k})$ and $\sigma^1(s) = s$ for all $s \in (s_1(k), \phi_1(\hat{k})]$, so any junior agent's value at $\phi_1(\hat{k})$ is:

$$V^1\phi_1(\hat{k})) = \tilde{\pi}^1\theta^1 - \pi^1 c(1-\delta)\phi_1(\hat{k}) + \tilde{\pi}^2\delta V^1(s_1(\hat{k})) \tag{A.7}$$

where $V^1(\phi_1(\hat{k}))$ must verify:

$$\delta V^1(\phi_1(\hat{k})) = \theta^1 - c(1-\delta)\tilde{s}(1,\hat{k})$$

and replacing in (A.7) we get

$$\phi_1(\hat{k}) = w^1(1)\left[1 + \frac{\delta\tilde{\pi}^2}{1-\delta}\frac{\theta^1 - \delta V^1(s_1(\hat{k}))}{\theta^1}\right] \tag{A.8}$$

(b) We then define[14]

$$\psi(k) := \frac{1 - \left(\delta\tilde{\pi}^2\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)\right)^k}{1-\delta\tilde{\pi}^2}\frac{\theta^2}{c}.$$

[14]Note that ψ is meant to disambiguate between the actual cutpoints $s_1(k)$, and those cutpoints of senior agents that would exist if there were arbitrarily many cutpoints below ϕ_1.

Of course, for all $k \leq \hat{k}$, $s_1(k) = \psi(k)$. Thus, if $\psi(\hat{k}+1) < \tilde{s}_1(1,\hat{k})$, then it *cannot* be that there exists an equilibrium which prescribes $\hat{k}$ cut-points for seniors below $\hat{\phi}_1(\hat{k})$. This is because while junior agents contribute to complete the project, the maximal amount any senior agent is willing to contribute to obtain the state $s_1(\hat{k})$ is $\psi(\hat{k}+1) - \psi(\hat{k}) < \phi_1(\hat{k}) - s_1(\hat{k})$. Thus, at $\phi_1(\hat{k})$, only a junior agent would be willing to contribute, contradicting the hypothesized equilibrium structure.

We now show precisely that $\phi_1(k) > \psi(k+1)$ for all $k \geq 1$. Consider first the case $k \geq 2$. From part *(*a),

$$\begin{aligned}\phi_1(k) &= \left[\frac{n^1}{n^1 - \delta\tilde{\pi}^1}\right]\frac{1}{c}\left\{\left[\frac{1-\delta\tilde{\pi}^1}{1-\delta}\right]\theta^1 - \left(\frac{\delta^2}{1-\delta}\right)\tilde{\pi}^2 V^1(s_1(k))\right\}\\ &= \frac{n^1}{n^1 - \delta\tilde{\pi}^1}\cdot\frac{1}{c}\left\{\theta^1 \cdot \left[\frac{1-(\delta\tilde{\pi}^2)^{k+1}}{1-\delta\tilde{\pi}^2}\right] + \theta^2 T(n^2)\right\}\end{aligned}$$

where

$$T(n^2) \equiv \frac{\delta^2\tilde{\pi}^1\tilde{\pi}^2\left(1-(\delta\tilde{\pi}^2)^k\left(n^2-(n^2-1)\left(1-\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)^k\right)\right)\right)}{n^1(1-\delta\tilde{\pi}^2)^2}$$

results from substituting $V^1(s_1(k))$ from (A.4) and simplifying. Noting that

$$\frac{\partial}{\partial n^2}\left(n^2-(n^2-1)\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)^k\right) = 1 + k\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)^{k-1} - (k+1)\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)^k,$$

and that for any k, the function $f(x) = k \cdot x^{k-1} - (k+1)x^k$ has its maximum at $x = \frac{k-1}{k+1}$ with $f(\frac{k-1}{k+1}) = \left(\frac{k-1}{k+1}\right)^{k-1}$, we see that

$$\frac{\partial}{\partial n^2}\left(n^2-(n^2-1)\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)^k\right) > 0.$$

Moreover, because

$$\lim_{n^2\to\infty}\left[n^2-(n^2-1)\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)^k\right] = (1-\delta\tilde{\pi}^2)k,$$

we see that

$$T(n^2) > \lim_{n^2\to\infty} T(n^2) = \frac{\delta^2\tilde{\pi}^1\tilde{\pi}^2\left(1-(\delta\tilde{\pi}^2)^k\left(1+(1-\delta\tilde{\pi}^2)\cdot k\right)\right)}{n^1(1-\delta\tilde{\pi}^2)^2}.$$

Now, $(\delta\tilde{\pi}^2)^k(1+k(1-\delta\tilde{\pi}^2))$ is decreasing in k. Hence,

$$T(n^2) > \frac{\delta^2\tilde{\pi}^1\tilde{\pi}^2\left(1-(\delta\tilde{\pi}^2)(2-\delta\tilde{\pi}^2)\right)}{n^1(1-\delta\tilde{\pi}^2)^2} > 0.$$

Therefore, for any $k \geq 2$,

$$\frac{\phi_1(k)}{\psi(k+1)} > \frac{n^1}{n^1 - \delta\tilde{\pi}^1} \cdot \frac{\theta^1}{\theta^2} \cdot \frac{1 - (\delta\tilde{\pi}^2)^{k+1}}{1 - \left(\delta\tilde{\pi}^2 \left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)\right)^{k+1}}.$$

Further using the fact that $w^1(1) > w^2(1) \implies \frac{n^1}{n^1-\delta\tilde{\pi}^1} \cdot \frac{\theta^1}{\theta^2} > \frac{n^2}{n^2-\delta\tilde{\pi}^1}$, we obtain

$$\begin{aligned}\frac{\phi_1(k)}{\psi(k+1)} &> \frac{n^2}{n^2 - \delta\tilde{\pi}^2} \cdot \frac{1 - (\delta\tilde{\pi}^2)^{k+1}}{1 - \left(\delta\tilde{\pi}^2 \left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)\right)^{k+1}} \\ &= \left(1 - (\delta\tilde{\pi}^2)^{k+1}\right) \cdot \left[\frac{n^2(n^2 - \delta\tilde{\pi}^2)^k}{(n^2 - \delta\tilde{\pi}^2)^{k+1} - (\delta\tilde{\pi}^2)^{k+1}(n^2-1)^{k+1}}\right] := R(n^2, k).\end{aligned}$$

For any fixed, arbitrary k, by direct inspection,

$$\text{sgn}\left(\frac{\partial}{\partial n^2} R(n^2, k)\right) = \text{sgn}\left(\left(n^2(k+1) - \delta\tilde{\pi}^2(kn^2+1)\right)\left(\delta\tilde{\pi}^2(n^2-1)\right)^k - \left(n^2 - \delta\tilde{\pi}^2\right)^{k+1}\right) = -1.$$

That is, $\frac{\partial}{\partial n^2} R(n^2, k) < 0$ so that from above,

$$\frac{\phi_1(k)}{\psi(k+1)} > R(n^2, k) > \lim_{n^2 \to \infty} R(n^2, k) = 1,$$

which yields the desired result $\phi_1(k) > \psi(k+1)$, for any $k \geq 2$.

Consider now the case where $k = 1$. From part *(a)* and step 2, we have that

$$\phi_1(1) = \frac{n^1(1 + \delta\tilde{\pi}^2)}{n^1 - \delta\tilde{\pi}_1} \frac{\theta^1}{c} + \frac{\delta^2\tilde{\pi}^1\tilde{\pi}^2 \cdot n^2}{(n^1 - \delta\tilde{\pi}^1)(n^2 - \delta\tilde{\pi}^2)} \frac{\theta^2}{c}$$

by direct computation. By definition,

$$\psi(2) = \frac{1 - (\delta\tilde{\pi}^2 \cdot \frac{n^2-1}{n^2-\delta\tilde{\pi}^2})^2}{1 - \delta\tilde{\pi}^2}, \text{ so that}$$

$$\begin{aligned}\frac{\phi_1(1)}{\psi(2)} &> \frac{n^1}{n^1 - \delta\tilde{\pi}^1} \cdot \frac{\theta^1}{\theta^2} \cdot \frac{1 + \delta\tilde{\pi}^2}{\frac{1-(\delta\tilde{\pi}^2 \cdot \frac{n^2-1}{n^2-\delta\tilde{\pi}^2})^2}{1-\delta\tilde{\pi}^2}} \\ &> \frac{n^2}{n^2 - \delta\tilde{\pi}^2} \cdot \frac{1 - (\delta\tilde{\pi}^2)^2}{1 - (\delta\tilde{\pi}^2 \cdot \frac{n^2-1}{n^2-\delta\tilde{\pi}^2})^2} \qquad (w_1(1) > w_2(1)) \\ &> \frac{n^2}{n^2 - \delta\tilde{\pi}^2} \cdot \frac{1 - (\delta\tilde{\pi}^2)^2}{(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2})^2(1 - (\delta\tilde{\pi}^2)^2)} \\ &= \frac{n^2(n^2 - \delta\tilde{\pi}^2)^2}{(n^2 - \delta\tilde{\pi}^2)(n^2-1)^2} > 1.\end{aligned}$$

Again, we obtain the desired result in this standalone, residual case. This establishes that there can be no eqbm. with finitely many senior cutpoints $\{s_1(k)\}_{k=1}^{\hat{k}}$ below ϕ_1, for any $\hat{k}$.

Step 6 (Conclusion). We now conclude the proof of Proposition 4.3. Step 5 rules out equilibria in which there are finitely many senior cutpoints. Thus, any equilibrium must have infinitely many cut-points with

$$\overline{s}_1 = \lim_{k\to\infty} s_1(k) = \lim_{k\to\infty} \frac{1-\left(\delta\tilde{\pi}^2\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)\right)^k}{1-\delta\tilde{\pi}^2}\frac{\theta^2}{c} = \left(\frac{1}{1-\delta\tilde{\pi}^2}\right)\frac{\theta^2}{c} < \phi_1.$$

Here, the second equality results from substituting the expression for $s_1(k)$ from (4.3), established in Step 2, and the final inequality is a direct consequence of Step 5. We now verify that the remaining candidate is, in fact, an equilibrium, and derive an explicit expression for ϕ_1. This completes the proof of the main proposition.

Step 3 uniquely identifies senior agent behavior as a cut-point strategy in equilibrium from any history h with $s^h < \overline{s}_1 < \phi_1$. Given the full completion ($\sigma^1(s) = s$) by junior agents from any $s \leq \phi_1$, it must be the case in any equilibrium— from Step 4— that $\sigma^2(s) = 0$ for all $s \in [\overline{s}_1, \phi_1]$. That is, the prescribed behavior of seniors is a best response to juniors' putative strategy $\sigma^1(s) = s$ for all $s \leq \phi_1$. It thus only remains to show there is no profitable deviation for juniors, given seniors' behavior.

Given that $\overline{s}_1 < \phi_1$, and $\sigma^2(s) = 0$ for all $s \in (\overline{s}_1, \phi_1]$, we have $V^1(\phi_1) = \tilde{\pi}^1\theta^1 + \tilde{\pi}^2\delta V^1(\phi_1) - \pi^1 c(1-\delta)\phi_1$ and $V^2(\phi_1) = \tilde{\pi}^1\theta^2 + \tilde{\pi}^2\delta V^2(\phi_1)$, and thus we have

$$V^1(\phi_1) = \left(\frac{1}{1-\delta\tilde{\pi}^2}\right)\left\{\tilde{\pi}^1\theta^1 - \pi^1 c(1-\delta)\phi_1\right\} \tag{A.9}$$

and

$$V^2(\phi_1) = \left(\frac{1-\tilde{\pi}^2}{1-\delta\tilde{\pi}^2}\right)\theta^2 \tag{A.10}$$

Since in equilibrium juniors' values must verify

$$\delta V^1(\phi_1) = \theta^1 - c(1-\delta)\phi_1,$$

(A.9) directly gives (4.2). Substituting in (A.9) and (A.10) gives (4.4).

Next we show that junior agents don't have a profitable deviation from any h with s^h in $[0, \phi_1]$. At a history h' with $s^{h'} = \phi_1$, by definition junior agents are indifferent between completing the project or staying put. They clearly cannot gain by moving to a non-cutpoint point, so it is enough to consider deviations to senior agents' cutpoints.

In equilibrium, $W^1(\phi_1, h') = \theta^1 - (1-\delta)c\phi_1$. Say the agent deviates and moves to some $s_1(k)$, contributing $e' = \phi_1 - s_1(k)$. By doing this, she gets a payoff:

$$W^1(e', h') = \delta V^1(s_1(k)) - (1-\delta)c(\phi_1 - s_1(k))$$

This is not a profitable deviation iff

$$0 \geq \delta V^1(s_1(k)) + (1-\delta)cs_1(k) - \theta^1 \equiv \mathbb{L}_k^1 \tag{A.11}$$

Using now (A.6) we have that

$$\frac{\mathbb{L}_k^1}{1-\delta} = -\theta^1 + \frac{\delta}{1-\delta}\tilde{\pi}^2\mathbb{L}_{k-1}^1 - \delta\tilde{\pi}^2 cs_1(k-1) + (1-\delta\pi^1)cs_1(k)$$

and replacing $s_1(k-1)$ with the expression from (A.5) we get

$$\begin{aligned}
\frac{\mathbb{L}_k^1}{1-\delta} &= -\theta^1 + \frac{\delta}{1-\delta}\tilde{\pi}^2\mathbb{L}_{k-1}^1 + cw^2(1)\frac{n^2-\delta\tilde{\pi}^2}{n^2-1} - cs_1(k)\left(\frac{1-\delta\tilde{\pi}^2}{n^2-1} + \delta\pi^1\right) \\
&\leq -\theta^1 + \frac{\delta}{1-\delta}\tilde{\pi}^2\mathbb{L}_{k-1}^1 + cw^2(1)\frac{n^2-\delta\tilde{\pi}^2}{n^2-1} - cw_2(1)\left(\frac{1-\delta\tilde{\pi}^2}{n^2-1} + \delta\pi^1\right) \\
&\leq \frac{\delta}{1-\delta}\tilde{\pi}^2\mathbb{L}_{k-1}^1 + c\left(1-\delta\pi^1\right)\left(w^2(1) - w^1(1)\right)
\end{aligned}$$

and the inequality is strict if $w^2(1) < s_1(k)$. Therefore it is necessary and sufficient for junior agents not to deviate to any $s_1(k)$ from arbitrary history h' with $s^{h'} = \phi_1$ if $\mathbb{L}_1^1 \leq 0$ which is true since

$$\begin{aligned}
\mathbb{L}_1^1 &= \delta\left(\theta^1 - \pi^1 c(1-\delta)w^2(1)\right) + (1-\delta)cw^2(1) - \theta^1 \\
&= (1-\delta)c(1-\delta\pi^1)(w^2(1) - w^1(1)) < 0.
\end{aligned}$$

□

Proof of Remark 4.5. For $\tau = 1$, $V^1(\phi_1) = (\alpha^1/\delta)\theta^1$, and for $2 \leq \tau \leq j(1)$:

$$V^1(\phi_\tau) = \alpha^1 V^1(\phi_{\tau-1}) \Rightarrow V^1(\phi_\tau) = (\alpha^1)^\tau \frac{1}{\delta}\theta^1$$

On the other hand, for $\tau : 2 \leq \tau \leq j(1)^*$:

$$\begin{aligned}
\phi_\tau - \phi_{\tau-1} &= \left(\frac{1}{1-\delta\tilde{\pi}^2 - \delta\tilde{\pi}^1/n^1}\right)\frac{\delta V^1(\phi_{\tau-1})}{c} \\
&= \phi_1(\alpha^1)^{\tau-1}
\end{aligned}$$

where we used (4.2) for $\tau = 1$. Summing up over all $2 \leq \tau \leq j(1)$ we get

$$\phi_{j(1)} = \phi_1\left[\sum_{\tau=0}^{j(1)-1}(\alpha^1)^\tau\right] = \phi_1\frac{1-(\alpha^1)^{j(1)}}{(1-\alpha^1)} \tag{A.12}$$

Thus, there is no alternation of effort on the equilibrium path iff $q < \phi_{j(1)}$, or

$$q < \phi_1 \frac{1-(\alpha^1)^{j(1)}}{1-\alpha^1} \quad (*)$$

□

Proof of Proposition 4.8. In the text we prove that if $\Delta^m < \Delta^{m'}$ then all stints of type m were one-step stints except for potentially the first one, which is the case only if $m' = 1$. We prove the result for $\Delta^2 < \Delta^1$. The case of $\Delta^2 > \Delta^1$ is analogous.

Note that for $k-1$ even, (4.14) gives

$$\Omega^2(\phi_{J(k-1)}) = \Delta^2\Omega^2(\phi_{J(k-2)}) \qquad \Delta^2 < \Omega^2(\phi_{J(k-1)}) < 1$$

so

$$\Delta^2 < \Omega^1(\phi_{H(k-2)}) < 1 \tag{A.13}$$

and for k odd, (4.14) gives

$$\Omega^1(\phi_{J(k)}) = [\Delta^1]^{j(k)}\Omega^1(\phi_{J(k-1)}) \qquad \Delta^1 < \Omega^1(\phi_{J(k)}) < 1$$

so

$$\Delta^1 < [\Delta^1]^{j(k)}\Omega^1(\phi_{J(k-1)}) < 1$$

or multiplying by Δ_2

$$\Delta^2\Delta^1 < [\Delta^1]^{j(k)}\Omega^1(\phi_{J(k-2)}) < \Delta^2 \tag{A.14}$$

Note that the left hand side of (A.14) and $\Omega^1(\phi_{J(k-2)}) < 1$ from (A.13) imply that $\Delta^2 < [\Delta^1]^{j(k)-1}$. Now consider $k-2$ which is odd and gives $\Delta^1 < \Omega^1(\phi_{J(k-2)}) < 1$. Thus,

$$\begin{aligned}
\Delta^1 &< \Delta^2\Omega^1(\phi_{J(k-1)})\\
[\Delta^1]^{j(k)+1} &< [\Delta^1]^{j(k)}\Delta^2\Omega^1(\phi_{J(k-1)})\\
[\Delta^1]^{j(k)+1} &< \Delta^2,
\end{aligned}$$

where the first inequality follows since $\Omega^2(\phi_{J(k-1)}) = \Delta^2\Omega^2(\phi_{J(k-2)})$, the second from multiplying by $[\Delta^1]^{j(k)}$, and the third from the fact that $[\Delta^1]^{j(k)}\Omega^1(\phi_{J(k-1)}) < 1$.

Note that it must be that for all k odd

$$\begin{aligned}
[\Delta^1]^{j(k)+1} &< \Delta^2 < [\Delta^1]^{j(k)-1}\\
j(k)-1 &< \frac{\log[\Delta^2]}{\log[\Delta^1]} < j(k)+1\\
\frac{\log[\Delta^2]}{\log[\Delta^1]} - 1 &< j(k) < \frac{\log[\Delta^2]}{\log[\Delta^1]} + 1
\end{aligned}$$

Assume first that $\log[\Delta^2] = \ell \log[\Delta^1]$ for some $\ell \in \mathbb{Z}^+$. Then we have that $\ell - 1 < j(k) < \ell + 1$ which implies that $j(k) = \ell$. Let now $\lfloor y \rfloor \equiv \sup\{x \in \mathbb{Z}_+ : x \leq y\}$, and assume that

$$\frac{\log[\Delta^2]}{\log[\Delta^1]} = \left\lfloor \frac{\log[\Delta^2]}{\log[\Delta^1]} \right\rfloor + \epsilon$$

for some $\epsilon \in (0,1)$. It follows then that

$$\left\lfloor \frac{\log[\Delta^2]}{\log[\Delta^1]} \right\rfloor - 1 + \epsilon < j(k) < \left\lfloor \frac{\log[\Delta^2]}{\log[\Delta^1]} \right\rfloor + 1 + \epsilon$$
$$\left\lfloor \frac{\log[\Delta^2]}{\log[\Delta^1]} \right\rfloor \leq j(k) \leq \left\lfloor \frac{\log[\Delta^2]}{\log[\Delta^1]} \right\rfloor + 1$$

□

Proof of Proposition 4.10. We show that the expected completion time of a project that requires L contribution stints, $\{h_1, \ldots, h_L\}$, is given by

$$\mathcal{E}\left(\{j(\ell)\}_{\ell=1}^{L}\right) = \sum_{\ell=1}^{L} j(\ell) + \sum_{\ell=1}^{L} \tilde{\pi}_{m_{-\ell}} (\tilde{\pi}_{m_\ell})^{j(\ell)-2}, \tag{A.15}$$

where $m_\ell = 1$ if ℓ is odd and 2 if ℓ is even, and $m_{-\ell} = m \in M \neq m_\ell$. The second part of the proposition follows by the argument in the text.

Consider the sequence $\{j(\ell)\}_{\ell=1}^{L}$ and define the random variable $x\left(\{j(\ell)\}_{\ell=1}^{L}\right)$, as the number of periods it takes to finish the project. Note that this random variable has the following distribution:

$$x\left(\{j(\ell)\}_{\ell=1}^{L}\right) = j(L) + x\left(\{h(\ell)\}_{\ell=1}^{L-1}\right) \qquad \text{w.p.} \qquad (\tilde{\pi}^1)^{j(L)}$$
$$x\left(\{j(\ell)\}_{\ell=1}^{L}\right) = j(L) + 1 + x\left(\{j(\ell)\}_{\ell=1}^{L-1}\right) \qquad \text{w.p.} \qquad (\tilde{\pi}^1)^{j(L)}\tilde{\pi}^2$$
$$\ldots$$
$$x\left(\{j(\ell)\}_{\ell=1}^{L}\right) = j(L) + k + x\left(\{j(\ell)\}_{\ell=1}^{L-1}\right) \qquad \text{w.p.} \qquad (\tilde{\pi}^1)^{j(L)}(\tilde{\pi}^2)^k$$

It follows that $x\left(\{j(\ell)\}_{\ell=1}^{L}\right) = x\left(j(L)\right) + x\left(\{j(\ell)\}_{\ell=1}^{L-1}\right) = \sum_{\ell=1}^{L} x\left(j(\ell)\right)$, and therefore

$$\mathbb{E}\left(x\left(\{j(\ell)\}_{\ell=1}^{L}\right)\right) = \sum_{\ell=1}^{L} \mathbb{E}\left(x\left(j(\ell)\right)\right)$$

Note that, letting $m_\ell = 1$ if ℓ is odd and 2 if ℓ is even, and $m_{-\ell} = m \in M \neq m_\ell$, we have:

$$
\begin{aligned}
\mathbb{E}\left(x\left(j(\ell)\right)\right) &= j(\ell) + (\tilde{\pi}_{m_\ell})^{j(\ell)} \left[\sum_{k=1}^{\infty} (\tilde{\pi}_{m_{-\ell}})^k k\right] \\
&= j(\ell) + (\tilde{\pi}_{m_\ell})^{j(\ell)-2} \left[\tilde{\pi}_{m_{-\ell}}\right],
\end{aligned}
$$

and therefore

$$
\mathbb{E}\left(x\left(\{j(\ell)\}_{\ell=1}^{L}\right)\right) = \sum_{\ell=1}^{L} j(\ell) + \sum_{\ell=1}^{L} (\tilde{\pi}_{m_\ell})^{j(\ell)-1} \left[\frac{\tilde{\pi}_{m_{-\ell}}}{\tilde{\pi}_{m_\ell}}\right]
$$

Now, suppose $q = \phi_\ell$ for $\ell > j(1)$ and $\Delta^m > \Delta^{m'}$, and let $\mathcal{E}_1 \equiv j(1) + \tilde{\pi}_2(\tilde{\pi}_1)^{j(1)-2}$, as in the statement of the proposition. Since $\Delta^m > \Delta^{m'}$, before the final stint of $j(1)$ small projects by juniors, juniors and seniors alternate in a contribution cycle in which group m' completes single small projects, while group m completes either $x^m \equiv \left\lfloor \frac{\log[\Delta^{m'}]}{\log[\Delta^m]} \right\rfloor$ or $x^m + 1$ small projects before turning it over to members of the other group. We can then split the terms of (A.15) in stints taken by each group, and incorporate this information to compute expected delay for a project of size q.

Suppose first that $\hat{k}(q)$ is odd. Then in addition to the last stint by group 1 agents there are $(\hat{k}(q) - 1)/2$ stints by each group, so for some $x \in [x^m, x^m + 1]$,

$$
\mathcal{E}\left(\{j(\ell)\}_{\ell=1}^{\hat{k}(q)}\right) = \mathcal{E}_1 + \left(\frac{\hat{k}(q) - 1}{2}\right) \left\{(1 + x) + \frac{\tilde{\pi}^m}{\tilde{\pi}^{m'}} + \tilde{\pi}^{m'} (\tilde{\pi}^m)^{x-2}\right\}.
$$

In particular, if $\Delta^m = \Delta^{m'}$, groups alternate in single step stints, so

$$
\mathcal{E}\left(\{j(\ell)\}_{\ell=1}^{\hat{k}(q)}\right) = \mathcal{E}_1 + \left(\hat{k}(q) - 1\right) \left\{1 + \frac{(\tilde{\pi}^m)^2 + (\tilde{\pi}^{m'})^2}{2\tilde{\pi}^{m'}\tilde{\pi}^m}\right\},
$$

Similarly, if $\hat{k}(q)$ is even, then in addition to the last stint by juniors there are $\hat{k}(q)/2$ stints by group m' agents and $\hat{k}(q)/2 - 1$ stints by group m agents, so that

$$
\mathcal{E}\left(\{j(\ell)\}_{\ell=1}^{\hat{k}(q)}\right) = \mathcal{E}_1 + \frac{\hat{k}(q)}{2} \left\{(1 + x) + \frac{\tilde{\pi}^m}{\tilde{\pi}^{m'}} + \tilde{\pi}^{m'} (\tilde{\pi}^m)^{x-2}\right\} - \left(x + \tilde{\pi}^{m'} (\tilde{\pi}^m)^{x-2}\right),
$$

□

Online Appendix

Proof of Lemma 4.1. (a) First we show that there exists a $b > 0$ such that $\sigma^m(h) = s^h$ for all m for all histories h such that $s^h \leq b$. Note that for $s^h \leq \min\{\theta^1, \theta^2\}/c$,

$$W^m(s^h, h) = \theta^m - (1-\delta)c \cdot s^h \geq \delta\theta^m, \ \forall m,$$

where the right-hand side is an upper bound on $W^m(e, h)$ for any $e < s^h$, since it is the value from zero contribution and the project being completed by another agent in the subsequent period. Letting $b = \min\{\theta^1, \theta^2\}/c$ completes the step.

(b) We now determine

$$w(1) \equiv \sup\{b : \sigma^m(s) = s \ \forall s \leq b, \forall m \in M\}.$$

It is clear that for any $s^h > \min_m\{\frac{\theta^m}{c(1-\delta)}\}$, we can never have $\sigma^m(h) = s^h$ for all m, for this would give some agents a negative payoff, making it profitable to deviate and contribute zero at s^h. Thus, there exists a finite $w(1)$.

Note from any h, that for $e < s^h$:

$$W^m(e, h) = \delta V^m(h_e) - (1-\delta)ce.$$

Take any h with $s^h \leq w(1)$. From (3.2), and since $\sigma^m(h') = s(h')$ for all m for histories h' with $s(h') < w(1)$, we have for any $e < s^h$,[15]

$$V^m(h_e) = \theta^m - \pi^m c(1-\delta)(s^h - e).$$

Thus for $e < s^h$:

$$W^m(e, h) = \delta\theta^m - c(1-\delta)\left(e + \delta\pi^m(s^h - e)\right).$$

Note that for any $0 < e < s^h < w(1)$:

$$W^m(0, h) - W^m(e, h) = c(1-\delta)e\left(1 - \delta\pi^m\right) > 0.$$

Thus, when all agents finish the project on their turn, an agent is better off contributing zero than contributing $e \in (0, s^h)$. Moreover, $W^m(s^h, h) \geq W^m(0, h)$ if and only if

$$s^h \leq \frac{\theta^m}{c\left(1 - \delta\pi^m\right)} \equiv w^m(1),$$

with equality uniquely at $s = w^m(1)$. Thus $\sigma^m(h) = s^h$ for $m = 1, 2$ if $s < w(1) \equiv \min_m w^m(1)$ and only if $s \leq w(1)$. In particular, from any history h with $s^h = w(1)$, we see it is always an equilibrium that $\sigma^m(h) = w(1)$ for $m = 1, 2$.

[15] Note that it is possible that $\sigma^m(h') \neq w(1)$ when $s(h') = w(1)$. However, in this case because $w(1)$ is the supremum of the states at which m contributes, she is indifferent between contributing and not. Hence, when $s = w(1)$ and $e = 0$, $V^m(h_e)$ will have the same value as that given in the above proof.

(c) We consider, moreover, conditions for the uniqueness of equilibrium behavior at histories h with corresponding state $w(1)$. Note that if $w^m(1) > w(1)$, then agents of type m have a strict incentive to contribute at such an h, and so we need only consider those types m for which $w^m(1) = w(1)$. From previous work, we know that $\sigma^m(h) \in \{0, w(1)\}$. Consider therefore an equilibrium in which $\sigma^m(h) = 0$. In such an equilibrium,

$$V^m(h) = \frac{\tilde{\pi}^{m'}\theta^m}{1-\delta\tilde{\pi}^m}. \tag{A.16}$$

Consider now a deviation of a player of type m to contributing $w(1)$ at h,

$$\begin{aligned} W^m(w(1), h) &= \theta^m - c(1-\delta)w(1) = \delta\theta^m \frac{1-\pi^m}{1-\delta\pi^m} \\ &= \delta V^m(h) \frac{\frac{1-\pi^m}{1-\delta\pi^m}}{\frac{1-\tilde{\pi}^m}{1-\delta\tilde{\pi}^m}} > \delta V^m(w(1)) \end{aligned} \tag{A.17}$$

Thus, there is a profitable deviation for the type m agents from the conjectured equilibrium and so we have that equilibrium behavior at any history with corresponding state $w(1)$ is unique and entails completion of the project.

(d) It remains to show that if $w^m(1) > w^{m'}(1)$, then $\sigma^m(h) = s^h$ for all h with $s^h \in (w^{m'}(1), w^m(1)]$. Consider first $s^h \in (w^{m'}(1), w^m(1))$. As before, $W^m(s^h, h) = \theta^m - c(1-\delta)s^h$. However, since type m' agents don't finish the project outright, for any $e \in [0, s^h)$ we have

$$W^m(e,h) < \delta\theta^m - c(1-\delta)\left[e + \delta\pi^m(s^h - e)\right] \leq \delta\left(\theta^m - c(1-\delta)\pi^m s^h\right)$$

Thus $W^m(s^h, h) > W^m(e, h)$ for all $e \in [0, s^h)$ if and only if $s^h < w^m(1)$. Now consider a history h for which $s^h = w^m(1)$. Using similar arguments we have that $W^m(w^m(1), h) > W^m(e, h)$ for all $e \in (0, w^m(1))$, so we only need to consider equilibria with $\sigma^m(h) \in \{0, w^m(1)\}$. The definition of $w^m(1)$ gives that $\sigma^m(h) = w^m(1)$ is indeed an equilibrium. For uniqueness, assume that $\sigma^m(h) = 0$ from some history h with corresponding state $w^m(1)$. It follows then that

$$V^m(h) = \frac{\tilde{\pi}^{m'}\delta}{1-\tilde{\pi}^m\delta} V^m\left(h_{\sigma^{m'}(h)})\right)$$

Since no type finishes the project outright, we must have that

$$V^m\left(h_{\sigma^{m'}(h)}\right) \leq \theta^m - \pi^m c(1-\delta)\left(w^m(1) - \sigma^{m'}(h)\right)$$

Therefore, there is a profitable deviation for type m if $\delta V^m(h) < \theta^m - c(1-\delta)w^m(1)$. For such a profitable deviation, it is sufficient that

$$\begin{aligned} \frac{\tilde{\pi}^{m'}\delta}{1-\tilde{\pi}^m\delta}\left(\theta^m - \pi^m c(1-\delta)\left(w^m(1) - \sigma^{m'}(h)\right)\right) &< \theta^m - c(1-\delta)w^m(1) \\ \frac{1-\tilde{\pi}^m}{n^m - \tilde{\pi}^m}\sigma^{m'}(h) &< w^m(1), \end{aligned}$$

which is true. ◻

Lemma A.1. *For all histories h with $s^h > \phi_1$ and $m \in \{1,2\}$, $\sigma^m(h) \leq s^h - \phi_1$.*

Proof of Lemma A.1. Consider first junior (group 1) agents. Note that by construction, ϕ_1 is the largest s such that from any h with $s^h = s$, juniors would want to finish the project outright when they anticipate that seniors do not contribute at h (i.e., given that $\sigma^2(h) = 0$). Since $\sigma^2(h) = 0$ minimizes free riding incentives, it follows that ϕ_1 is the largest state corresponding to any history from which juniors would want to finish the project outright, for any σ^2. Thus, at any history h with $s^h > \phi_1$, $\sigma^1(h) < s^h$. Now, we know by Lemma 4.1 that from any h with $s^h \in (0, \phi_1)$, a junior agent strictly prefers to finish the project outright than playing $e \in [0, s^h)$. By the linearity of costs, it follows that for h with $s^h > \phi_1$ a junior agent would weakly prefer to finish the project outright than to move the project to $s' \in (0, \phi_1)$. Thus it follows that at any history h with $s^h > \phi_1$, $\sigma^1(h) \leq s^h - \phi_1$.

Next consider senior agents. Since σ^2 is a cutpoint strategy in $[0, \left(\frac{1}{1-\delta\tilde{\pi}^2}\right)\frac{\theta^2}{c})$, we know that for any k in the sequence, and any $s > s_1(k)$, $\sigma^2(s) \leq s - s_1(k)$. Thus, for all $s > \phi_1$, $\sigma^2(s) \leq s - \left(\frac{1}{1-\delta\tilde{\pi}^2}\right)\frac{\theta^2}{c}$. Now, we know that in equilibrium $\sigma^2(s) = 0$ for all $s \in [\left(\frac{1}{1-\delta\tilde{\pi}^2}\right)\frac{\theta^2}{c}), \phi_1]$. It follows immediately that for any history h with $s^h > \phi_1$, moving the project to $s' \in [\left(\frac{1}{1-\delta\tilde{\pi}^2}\right)\frac{\theta^2}{c}), \phi_1)$ is dominated by moving the project to ϕ_1, since this doesn't increase delay and reduces expected costs. Thus, for any h with $s^h > \phi_1$, $\sigma^2(h) \leq s^h - \phi_1$. ◻

Proof of Theorem 4.9. We prove the theorem for the case of public goods with an infinite-stream of payoffs θ^m in each period and then analogize to the case of a finite stream of payoffs of length T, which allows us to make the efficiency comparison as $\delta \to 1$. In particular, we prove the following result:

1. *Suppose $(\Delta^1)^y < \Delta^2 < \Delta^1$ for some $y \in \mathbb{N}$. Let y^* denote the smallest integer y^* such that this inequality holds. Then,*

$$\tilde{\phi} \leq \frac{\theta^1}{c(1-\delta\tilde{\pi}^2-\pi_1)}\left[\frac{1-(\alpha^1)^{j(1)}}{1-\alpha^1} + \frac{(\alpha^1)^{j(1)}\beta^1(1-(\alpha^1)^{y^*})}{(1-\alpha^1)(1-\beta^1(\alpha^1)^{y^*})}\right] + \frac{(\beta^2)^{j(1)}\theta^2}{c(1-\delta\tilde{\pi}^1-\delta\pi_2)(1-\alpha^2(\beta^2)^{y^*})}.$$

2. *Suppose $(\Delta^2)^y < \Delta^1 < \Delta^2$ for some $y \in \mathbb{N}$. Let y^* denote the smallest integer y^* such that this inequality holds. Then,*

$$\tilde{\phi} \leq \frac{\theta^1}{c(1-\delta\tilde{\pi}^2-\pi_1)}\left[\frac{1-(\alpha^1)^{j(1)}}{1-\alpha^1} + \frac{(\alpha^1)^{j(1)}(\beta^1)^{y^*}}{1-\alpha^1(\beta^1)^{y^*}}\right] + \frac{(\beta^2)^{j(1)}(1-(\alpha^2)^{y^*})\theta^2}{c(1-\delta\tilde{\pi}^1-\delta\pi_2)(1-\alpha^2)(1-\beta^2(\alpha^2)^{y^*})}.$$

3. *Suppose $\Delta_1 = \Delta_2 = \Delta$. Then*

$$\tilde{\phi} \leq \frac{\theta^1}{c(1-\delta\tilde{\pi}^2-\pi_1)}\left[\frac{1-(\alpha^1)^{j(1)}}{1-\alpha^1} + \frac{(\alpha^1)^{j(1)}\beta^1}{(1-\beta^1\alpha^1)}\right] + \frac{(\beta^2)^{j(1)}\theta^2}{c(1-\delta\tilde{\pi}^1-\delta\pi_2)(1-\alpha^2\beta^2)}.$$

Proof. We prove (1), as (2) is analogous. Part (3) follows directly from (1) or (2), making $y^* = 1$. From (4.2), we have

$$
\begin{aligned}
\phi_1 &= \left(\frac{1}{1 - \delta\tilde{\pi}^2 - \delta\pi^1}\right)\frac{\theta^1}{c} \\
\phi_2 &= \phi_1 + \left(\frac{1}{1 - \delta\tilde{\pi}^2 - \delta\pi^1}\right)\frac{\delta V^1(\phi_1)}{c} \\
&= \left(\frac{1}{1 - \delta\tilde{\pi}^2 - \delta\pi^1}\right)\frac{(1+\alpha^1)\theta^1}{c} \\
&\vdots \\
\phi_{j(1)} &= \left(\frac{1}{1 - \delta\tilde{\pi}^2 - \delta\pi^1}\right)\frac{\left(1+\alpha^1+\ldots+\alpha^{h_1-1}\right)\theta^1}{c} \\
&= \left(\frac{1}{1 - \delta\tilde{\pi}^2 - \delta\pi_1}\right)\left(\frac{1-(\alpha^1)^{h_1}}{1-\alpha^1}\right)\frac{\theta^1}{c}.
\end{aligned}
$$

At $\phi_{j(1)+1}$, the active contributor switches from juniors (group 1) to seniors (group 2). Hence,

$$
\begin{aligned}
\phi_{j(1)+1} &= \phi_{j(1)} + \left(\frac{1}{1 - \delta\tilde{\pi}^1 - \delta\pi^2}\right)\frac{\delta V^2(\phi_{j(1)})}{c} \\
&= \phi_{j(1)} + \left(\frac{1}{1 - \delta\tilde{\pi}^1 - \delta\pi^2}\right)\frac{(\beta^2)^{j(1)}\theta^2}{c}.
\end{aligned}
$$

In the case (1) being analyzed, $J(2) = j(1) + 1$ since each stint of senior agents is of length one. Continuing forward,

$$
\phi_{J(3)} = \phi_{J(2)} + \left(\frac{1}{1 - \delta\tilde{\pi}^2 - \delta\pi^1}\right)\frac{(\alpha^1)^{j(1)}\beta^1\theta^1}{c}\left(\frac{1-(\alpha^1)^{j(3)}}{1-\alpha^1}\right),
$$

analogous to the formulation of $\phi_{j(1)}$. Moreover, note that by case (1), $j(3) \leq y^*$, and substituting gives us

$$
\phi_{J(3)} \leq \phi_{J(1)} + \left(\frac{1}{1 - \delta\tilde{\pi}^1 - \delta\pi^2}\right)\frac{(\beta^2)^{j(1)}\theta^2}{c} + \left(\frac{1}{1 - \delta\tilde{\pi}^2 - \delta\pi^1}\right)\frac{(\alpha^1)^{j(1)}\beta^1\theta^1}{c}\left(\frac{1-(\alpha^1)^{y^*}}{1-\alpha^1}\right).
$$

In general, note that the Lth stint for $L > 1$ odd will consist of at most y^* steps actively contributed by juniors, and the $(L+1)-$th stint one step actively contributed by senior agents. With this emergent pattern, for any $L \geq 1$, the following formula holds for odd numbered stints:

$$\phi_{J(2L+1)} \leq \phi_{J(1)} + \left(\frac{1}{1-\delta\tilde{\pi}^1-\delta\pi^2}\right)\frac{(\beta^2)^{j(1)}\theta^2}{c}\sum_{k=0}^{L-1}\left(\alpha^2(\beta^2)^{y^*}\right)^k +$$
$$\left(\frac{1}{1-\delta\tilde{\pi}^2-\delta\pi^1}\right)\frac{(\alpha^1)^{j(1)}\beta^1\theta^1}{c}\left(\frac{1-(\alpha^1)^{y^*}}{1-\alpha^1}\right)\sum_{k=0}^{L-1}\left(\beta^1(\alpha^1)^{y^*}\right)^k. \quad \text{(A.18)}$$

Substituting for $\phi_{J(1)}$ from above and taking the limit as L goes to infinity in (A.18) gives us

$$\tilde{\phi} \leq \frac{\theta^1}{c(1-\delta\tilde{\pi}^2-\pi^1)}\left[\frac{1-(\alpha^1)^{j(1)}}{1-\alpha^1}+\frac{(\alpha^1)^{j(1)}\beta^1(1-(\alpha^1)^{y^*})}{(1-\alpha^1)(1-\beta^1(\alpha^1)^{y^*})}\right]$$
$$+\frac{(\beta^2)^{j(1)}\theta^2}{c(1-\delta\tilde{\pi}^1-\delta\pi^2)(1-\alpha^2(\beta^2)^{y^*})} \quad \text{(A.19)}$$

We consider two possible cases. Suppose first that there exists minimal positive integer y^* such that $(\Delta^1)^{y^*} < \Delta^2 < \Delta^1$. From (A.19),

$$\tilde{\phi} \leq \frac{\theta^1}{c(1-\delta\tilde{\pi}^2-\delta\pi^1)}\left[\frac{1}{1-\alpha^1}-\frac{(\alpha^1)^{j(1)}(1-\beta^1)}{(1-\alpha^1)(1-\beta^1(\alpha^1)^{y^*})}\right]+\frac{(\beta^2)^{j(1)}\theta^2}{c(1-\delta\tilde{\pi}^1-\delta\pi^2)(1-\alpha^2(\beta^2)^{y^*})}$$
$$< \frac{\theta^1}{c(1-\delta\tilde{\pi}^2-\delta\pi^1)(1-\alpha^1)}+\frac{\theta^2}{c(1-\delta\tilde{\pi}^1-\delta\pi^2)(1-\alpha^2)},$$

where the second inequality follows from the fact that $\beta^2 < 1$. Furthermore, for each group $m \in \{1, 2\}$,

$$1-\alpha^m = 1-\frac{(1-1/n^m)\tilde{\pi}^m\delta}{1-\delta\tilde{\pi}^{-m}-\delta\pi^m} = \frac{1-\delta}{1-\delta\tilde{\pi}^m-\delta\pi^m}$$

Substituting into the above inequality, and re-arranging we obtain

$$c\tilde{\phi} < \frac{\theta^1+\theta^2}{1-\delta}$$

for any fixed $\delta < 1$, in the case of infinte-lasting flow payoffs. For a project whose flow payoffs last exactly T periods, the analogous expression is

$$c\tilde{\phi} < \frac{(1-\delta^T)(\theta^1+\theta^2)}{1-\delta} = \hat{\theta}^1+\hat{\theta}^2.$$

If instead there does not exist a minimal positive integer y^* such that $(\Delta^1)^{y^*} < \Delta^2 < \Delta^1$, there exists a positive integer y^* such that $(\Delta^2)^{y^*} < \Delta^1 < \Delta^2$. From (A.19),

$$\begin{aligned}\tilde{\phi} &\leq \frac{\theta^1}{c(1-\delta\tilde{\pi}^2-\delta\pi^1)}\left[\frac{1-(\alpha^1)^{j(1)}}{1-\alpha^1}+\frac{(\alpha^1)^{j(1)}(\beta^1)^{y^*}}{1-\alpha^1(\beta^1)^{y^*}}\right]+\frac{(\beta^2)^{j(1)}(1-(\alpha^2)^{y^*})\theta^2}{c(1-\delta\tilde{\pi}^1-\delta\pi^2)(1-\alpha^2)(1-\beta^2(\alpha^2)^{y^*})} \\ &< \frac{\theta^1}{c(1-\delta\tilde{\pi}^2-\delta\pi^1)(1-\alpha^1)}+\frac{\theta^2}{c(1-\delta\tilde{\pi}^1-\delta\pi^2)(1-\alpha^2)},\end{aligned}$$

so that we again obtain the first desired result,

$$c\tilde{\phi} \leq \frac{\theta^1+\theta^2}{1-\delta}.$$

Redacting the flow payoffs from the public good to a horizon of length T gives the analogous expression

$$c\tilde{\phi} \leq \hat{\theta}^1+\hat{\theta}^2.$$

Since for any $\delta < 1$, we have $c\tilde{\phi} < \hat{\theta}^1+\hat{\theta}^2$, then for $n^m > 1$ for some $m \in \{1,2\}$,

$$\lim_{\delta\to 1} c\tilde{\phi} \leq \hat{\theta}^1+\hat{\theta}^2 < n^1\hat{\theta}^1+n^2\hat{\theta}^2.$$

Since $\tilde{\phi}$ is the maximal quantity q corresponding to projects completed in equilibrium, this gives us the second result.

□